Unitary groups and uniformisation of curves over a local field (preliminary version)

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Abstract

Let K be a complete nonarchimedean local field of characteristic zero having a residue field of odd characteristic. Let $L \supset K$ be the unramified quadratic extension and let ℓ be its residue field. For a discrete co-compact subgroup Γ of the unitary group $PU(3, L)$, a one-dimensional rigid analytic space on which Γ acts discontinuously is constructed. The quotients of this space by normal subgroups $\Delta \subset \Gamma$ of finite index and without elements of finite order are projective algebraic curves, on which the finite group Γ/Δ acts. The reduction of the space consists of hermitian curves and projective lines intersecting in ℓ-valued points. [1]

Introduction

Let $L \supset K$ is the unramified quadratic extension of a local field K of characteristic zero with residue field k of odd order q. We construct a uniformisation of curves by discrete co-compact subgroups Γ of the unitary group $PU(3, L)$. The construction has two ingredients. First a Γ-invariant subset of the affine

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building $\mathcal{B}$ of the group $PU(3, L)$ is determined. Then this Γ-invariant subset is used to construct a one-dimensional uniformising space on which Γ acts discretely with proper quotients.

The building $\mathcal{B}$ of the group $PU(3, L)$ is a $(q^3 + 1, q + 1)$-tree. The Γ-invariant subset is a union of $(q + 1, q + 1)$-subtrees that are subbuildings belonging to subgroups $PU(2, L)$ and that intersect pairwise in at most a vertex. We require that this subset contains all the vertices that are contained in $q^3 + 1$ edges. If the Γ-invariant subset of the building differs from $\mathcal{B}$, then the complement consists of isolated vertices.

The uniformising space is constructed by associating to each $PU(2, L)$-building in our Γ-invariant set a rigid analytic variety. This rigid analytic variety is choosen in such a way that $\Gamma \bigcap P(U(1, L) \times U(2, L))$ acts on it discretely with proper quotients. Here $P(U(1, L) \times U(2, L)) \subset PU(3, L)$ is the stabiliser of the subbuilding. We take a suitable open admissable subspace of each of these rigid varieties and glue them according to the intersections of the $PU(2, L)$-buildings in our Γ-invariant subset of $\mathcal{B}$. For the glueing to be possible, it is necessary that the group $PU(3, \ell)$ acts on the components of the reduction of the analytical space belonging to a $PU(2, L)$-building that correspond to vertices contained in $q^3 + 1$ edges of $\mathcal{B}$. Here ℓ denotes the residue field of L. Finally, a component has to be added for each vertex in $\mathcal{B}$ that is not contained in the Γ-invariant subset. The analytic space we associate to a subbuilding belonging to a group $PU(2, L)$ in our construction is an étale covering of the p-adic upper halfplane Ω_1. Over a suitable field extension this covering is isomorphic to a connected component of Drinfelds étale covering of the p-adic upper halfplane. The components of the reduction of this space are hermitian curves.

Let us briefly compare our construction with the uniformisation of curves over the field of complex numbers $\mathbb{C}$ by discrete subgroups with finite co-volume of $SL(2, \mathbb{R})$. Then the uniformising space is a hermitian symmetric space. This is the complex unit ball. The quotient of this symmetric space by the discrete group is either compact or non-compact, depending on whether the discrete group is co-compact or has only finite co-volume. If the quotient is non-compact it can always be compactified.

Over the completion of the algebraic closure $\mathbb{C}_p$ of the field K every algebraic curve with bad reduction has a uniformisation as a quotient of a suitable choosen rigid analytic variety by a discrete subgroup of $PGL(2, \mathbb{C}_p)$. The discrete group acts on a tree. Since the building of $PU(3, L)$ is also a tree, it does not seem unreasonable to expect for discrete subgroups of

$PU(3,L)$ to act discretely on a rigid analytic variety of dimension one with proper quotients.

If the characteristic of the local field is zero, then all discrete subgroups with finite co-volume of the linear algebraic group are co-compact. Therefore the idea of compactifying some symmetric space as in the real case seems at first absurd. We make sense of this situation by reverse engineering the notion of compactification. First we construct a Γ-invariant "open" subspace of the building $\mathcal{B}$ that omits a Γ-invariant set of isolated vertices. The "compactification" of this "open" subspace is obtained by adding the missing vertices and equals the building itself. Then we construct a suitable analytic variety for the open subspace of the building on which the group Γ acts discontinuously. Finally, this analytic space is compactified by adding suitable affinoid spaces for the vertices of the building that are missing in the open subspace.

This construction, however, does not give a bijection between "open" subspaces of the building $\mathcal{B}$ and commensurability classes of discrete co-compact subgroups of $PU(3,L)$. A discrete co-compact subgroup $\Gamma \subset PU(3,L)$ stabilises at most finitely many (possibly zero) suitable "open" subspaces of the building. On the other hand a suitable "open" subset of the building is stabilised by at most one commensurability class of discrete co-compact subgroups of $PU(3,L)$. Almost all arithmetic subgroups of $PU(3,L)$ preserve some suitable "open" subspace of the building.

We now give a brief outline of the article. The first four sections study the building $\mathcal{B}$ of $PU(3,L)$ and the action of discrete groups on $\mathcal{B}$. In §1 we recall some basic definitions and properties of the group $PU(3,L)$ and its building $\mathcal{B}$. In §2 we discuss (partial) spreads of the set of isotropic points for the unitary form over the residue field ℓ of the field L. Using such (partial) spreads, we define coverings by subbuildings of the building of the group $PU(3,\ell)$ over the residue field. In §3 we show that $\mathcal{B}$ can be covered by $PU(2,L)$ subbuildings in such a way that two such $PU(2,L)$ buildings intersect in at most a vertex. We give a precise definition of the type of coverings of $\mathcal{B}$ by $PU(2,L)$-buildings we need. We show that the automorphism group of such a covering of $\mathcal{B}$ is a discrete subgroup of $PU(3,L)$.

In §4 it is shown that almost all arithmetic groups $\Gamma \subset PU(3,L)$, preserve a Γ-invariant subset of $\mathcal{B}$ that is covered by $PU(2,L)$-buildings having the required properties.

In sections §5 till §6 we define and study a finite étale covering of the p-adic upper halfplane Ω_1 in detail. In §5 we construct the étale covering Σ

over the field L by glueing affinoids.

In §6 the covering is embedded into a projective plane. To do this the space Σ over the field L is constructed by glueing open admissable subsets of Σ together. These admissable subsets are embedded in two projective planes on which the group $P(U(1,L) \times U(2,L))$ acts linearly. The embeddings depend on a discrete co-compact subgroup $\Gamma_{\mathbf{b}} \subset P(U(1,L) \times U(2,L))$ and are invariant under the action of this subgroup. Then Σ is obtained by glueing these admissable subsets. As a result one has locally defined coordinates on Σ such that the discrete group $\Gamma_{\mathbf{b}} \subset P(U(1,L) \times U(2,L))$ acts linearly.

In §7 we recall some properties of the set $Y^s \subset \mathbb{P}^2_L$ consisting of the points in $\mathbb{P}^2_L$ that are stable for all maximal K-split tori in $PU(3,L)$. In particular, we define a $PU(3,L)$-equivariant map from the set Y^s to the building $\mathcal{B}$ of the group $PU(3,L)$.

This map is used in §8 to define an open admissable subspace $\Sigma^\circ \subset \Sigma$. The spaces Σ° for suitable subgroups $P(U(1,L) \times U(2,L)) \subset PU(3,L)$ can be glued together to form a space on which a co-compact discrete group $\Gamma \subset PU(3,L)$ acts discontinuously. This is done in §9. The quotient by the group Γ is not always complete, but it can be compactified. The construction of the uniformising space is analogous to the construction of Σ by glueing together admissable subspaces. In §10 the space is compactified Γ-equivariantly.

In §11 we describe the reduction, a semistable reduction and determine the genus of the quotients. In §12 we show that our variety differs from certain known moduli spaces. We also speculate on a possible generalisation of the construction to other groups.

In §13 we discuss some examples of spreads, discrete groups and algebraic curves.

Contents

1 The group $PU(3, L)$ and its building

1.1. The field. Let $L \supset K$ be as above. We write K° (L°) for the ring of integers of K (L). The residue fields are denoted by k and ℓ. By q we denote the number of elements in the residue field of K. Then q is some power of $p := char(k) > 2$. Let v be the additive valuation on L, normalised such that $v(L^*) = \mathbb{Z}$. The absolute value of x in L is $|x| := q^{-2v(x)}$. We fix an uniformizer π in L°, $v(\pi) = 1$.

1.2. The unitary groups. Let $V \cong L^3$ be a vector space equiped with a non-degenerated unitary form h. Then there exists an L-basis $e_1,\ e_0,\ e_2$ of V, such that h has the standard form $h(x, y) = x_1\overline{y}_2 + x_2\overline{y}_1 + x_0\overline{y}_0$. Here $x_1,\ x_0,\ x_2$ are the coordinates of V (or $\mathbb{P}^2_L := \mathbb{P}(V)$) with respect to the basis e_1, e_0, e_2 and $^-$ denotes the action of the nontrivial element of the Galois group $Gal(L/K)$. The image of a vector $v \in V$ in $\mathbb{P}(V)$ will be denoted by $[v]$. By $U(3, L)$ we will mean the group of three by three matrices with coefficients in L that act on V and preserve the form h. The subgroup $SU(3, L) \subset U(3, L)$ consists of the elements having determinant one. The quotients of these groups by diagonal elements are denoted by $PU(3,L)$ and $PSU(3,L)$, respectively. Occasionally we will view these groups as a group $G(K)$ of K-rational points of a linear algebraic group G defined over K.

1.3. The building. In the L-module V we introduce the two L^0-submodules $M_0 :=< e_0, e_1, e_2 >$ and $M_1 :=< e_0, \pi e_1, e_2 >$. For a L^0-submodule M of V, we write [M] for the equivalence class $\{\lambda \cdot M | \lambda \in L^*\}$.

The building $\mathcal{B}$ of $PU(3,L)$ is the tree, whose vertices are given by the $PU(3,L)$ images of $[M_0]$ and $[M_1]$. The edges (or chambers) are given by the $PU(3,L)$ images of $\{[M_0],[M_1]\}$. Since L/K is unramified, a vertex of type $g([M_0])$ with $g \in SU(3,L)$ is contained in q^3+1 edges while the other vertices are contained in $q+1$ edges. Both type of vertices are special, i. e. they are stabilized by the full Weyl group of the root system. (See [A-B] def. 10.18.) We have a type map τ that associates to a vertex $\mathbf{v} \in \mathcal{B}$ its type $\tau(\mathbf{v}) \in \{0,1\}$. The vertices $\mathbf{v}$ corresponding to equivalence classes $g([M_0])$, $g \in PU(3,L)$, are of type $\tau(\mathbf{v}) = 0$ and are called *hyperspecial*. They remain special for any unramified extension of the field K. (See [Ti] §1.10.) The vertices $\mathbf{v}$ corresponding to equivalence classes $g([M_1])$ have type $\tau(\mathbf{v}) = 1$.

Let $S \subset G(K)$ be a maximal K-split torus. We may assume that S is the torus that acts diagonally with respect to the basis e_0, e_1, e_2 of V. Let $A \subset \mathcal{B}$ be the apartment associated to the maximal K-split torus $S(K) \cong K^*$. Then S acts on A by translations. The vertices of A are $[M_n], n \in \mathbb{Z}$ where $M_{2n} =< e_0, \pi^n e_1, \pi^{-n} e_2 >$ and $M_{2n+1} =< e_0, \pi^{n+1} e_1, \pi^{-n} e_2 >$. We will often identify the apartment A with the real line $\mathbb{R}$. This will always be done in such a way that the vertices correspond to the set of integers. We will then write n for the vertex $[M_n]$. This identification of the apartment A with the real line $\mathbb{R}$ gives a distance $d_{\mathcal{B}}(a,b) := |a-b|$ on A. This distance can be extended to the entire building $\mathcal{B}$.

1.4. Embedding the building into the building of $PGL(3,L)$. The group $PU(3,L)$ is a subgroup $PGL(3,L)$ fixed by an involution. Therefore the building $\mathcal{B}$ of the group $PU(3,L)$ can be obtained as the set of points fixed by an involution acting on the building $\mathbb{B}$ of the group $PGL(3,L)$.

Let us first recall the description of the building $\mathbb{B}$ in terms of equivalence classes of L°-modules. The vertices of $\mathbb{B}$ are given by the equivalence classes of L°-modules in the L-module $V \cong L^3$. The maximal simplices or chambers in the building are triangles. Three vertices $\mathbf{v}_0, \mathbf{v}_1, \mathbf{v}_2 \in \mathbb{B}$ form a chamber if and only if the corresponding equivalence classes of L°-modules can be represented by modules $M_{\mathbf{v}_i},\ i = 1, 2, 3$ that satisfy: $\pi M_{\mathbf{v}_0} \subset M_{\mathbf{v}_1} \subset M_{\mathbf{v}_2} \subset M_{\mathbf{v}_0}$.

Let $M_0 := < e_0, e_1, e_2 >$, $M_1 := < e_0, \pi e_1, e_2 >$ and $M_2 := < \pi e_0, \pi e_1, e_2 >$ be three L°-modules. Then $[M_0], [M_1], [M_2]$ define a triangle in the building $\mathbb{B}$.

Using the unitary form h on $V \cong L^3$ one defines the dual $M^\vee$ of a L°-module M as being the L°-module $M^\vee := \{x \in V \mid \forall (y \in M)\ h(x, y) \in L^\circ\}$. Then the map $M \to M^\vee$ induces an involution τ on the vertices of the building $\mathbb{B}$. The map τ can be extended to the entire building $\mathbb{B}$. Let $\mathbb{B}^\tau \subset \mathbb{B}$ be the set of points that are fixed by the involution τ. Then $\mathbb{B}^\tau$ inherits a simplicial structure from the building $\mathbb{B}$. The simplices of $\mathbb{B}^\tau$ are the intersections of $\mathbb{B}^\tau$ with simplices of the building $\mathbb{B}$ that are non-empty. In fact $\mathbb{B}^\tau$ with this simplicial structure is the building $\mathcal{B}$ of the group $PU(3, L)$.

1.5. The building of the group $PU(3, \ell)$. Let $\ell \cong \mathbb{F}_{q^2}$ be the residue field of L. Let the vector space ℓ^3 be equiped with a non-degenerated unitary form $h_\ell(x, y)$. The group $PU(3, \ell)$ acts on the space ℓ^3 and on $\mathbb{P}^2_\ell$ preserving the unitary form $h_\ell(x, y)$. The group preserves the hermitian curve $\mathcal{H} \subset \mathbb{P}^2_\ell$ defined by $h_\ell(x, x) = 0$. The ℓ-valued isotropic points in $\mathbb{P}^2_\ell$ form a subset $\mathcal{H}(\ell) \subset \mathbb{P}^2(\ell)$ of order $\sharp\mathcal{H}(\ell) = q^3 + 1$.

The vertices of the building of the group $PU(3, \ell)$ correspond to the isotropic points of $\mathbb{P}^2(\ell)$. Two such isotropic points define an apartment of the building.

The building can also be defined as the *link* of a hyperspecial vertex $\mathbf{v} \in \mathcal{B}$. (See [A-B] definition A.19 and proposition 4.9.) The link $lk_{\mathcal{B}}(\mathbf{v})$ of a vertex $\mathbf{v} \in \mathcal{B}$ consists of the vertices $\mathbf{v}' \in \mathcal{B}$ that form an edge $\mathbf{e}'$ with $\mathbf{v}$.

Let $[M_{\mathbf{v}}]$ be the equivalence class of L°-modules belonging to the hyperspecial vertex $\mathbf{v}$. The building $lk_{\mathcal{B}}(\mathbf{v})$ is the building of the group $PU(3, \ell)$ that acts on $\mathbb{P}^2_L = \mathbb{P}(M_{\mathbf{v}} \otimes \ell)$ preserving the unitary form $h \otimes \ell$.

The link $lk_{\mathcal{B}}(\mathbf{v}_1)$ for a non-hyperspecial vertex $\mathbf{v}_1 \in \mathcal{B}$ is isomorphic to the building of a group $PU(2, \ell)$.

2 Spreads and buildings

Spreads are partitions of the set of isotropic points in $\mathbb{P}^2_\ell$ into subsets of $q+1$ isotropic points. We use partial spreads to define partial coverings of the building of the group $PU(3, \ell)$ by buildings belonging to subgroups $SU(2, \ell)$.

2.1 Definition. A *spread* of the ℓ-valued isotropic points in $\mathbb{P}^2_\ell$ for a non-degenerated hermitian form $h_\ell(x,y)$ consists of a partition of the set of isotropic points into lines $\mathbb{P}^1_\ell$ that each contain $q+1$ isotropic points. Since $\mathbb{P}(\ell)$ contains q^3+1 isotropic points, a spread consists of q^2-q+1 distinct lines. A *partial spread* consists of a non-empty set of lines $\mathbb{P}^1_\ell$ that does not cover the entire pointset and such that no isotropic point is contained in more than one line. Each line contains $q+1$ isotropic points.

In [B-E-K-S] different spreads of the hermitian curve are studied. Moreover, they give examples of partial spreads that are maximal (See [B-E-K-S] theorems 4.1 and 4.2.). However, they have an extra condition on these partial spreads. So it might be that they can be extended if one drops this extra condition. These results are also discussed in [B-E] §6.2.

2.2 Example. Let $x \in \mathbb{P}^2_\ell$ be an ℓ-valued anisotropic point and let $x^\perp \subset \mathbb{P}^2_\ell$ be the line orthogonal to the point x. Then the lines $\mathbb{P}^1_\ell$ orthogonal to the ℓ-valued anisotropic points contained in the line $x^\perp$ together with the line $x^\perp$ form a spread. The group $P(U(1,\ell)\times U(2,\ell))$ that stabilizes x and $x^\perp$ preserves the spread.

In propositions 13.3 and 13.4 we describe the spreads that are invariant under the groups $S_3 \ltimes C^2_{q+1}$ and C^2_{q+1}.

2.3 Definition. Let $G_0 \subset PU(3,\ell)$ be a finite group. A G_0-invariant partial spread is called G_0*-adapted* if every isotropic point with non-trivial stabiliser in G_0 is contained in the union of the spread.

2.4 Example. *A Singer cycle* C_{q^2-q+1}. Let $G_0 \subset PU(3,\ell)$ be a cyclic group of order q^2-q+1. Then G_0 is called a Coxeter torus or Singer cycle of the unitary group. The group G_0 has $q+1$ orbits each consisting of q^2-q+1 points on the set of ℓ-valued isotropic points. In particular, if a G_0-invariant spread exists, it would consist of a G_0-orbit of a single line $\mathbb{P}^1_\ell$. Such a cyclic spread does not exist for $p>2$. (See [B-E-K-S] theorem 3.1.)

2.5 Definition. If the hermitian form is non-degenerated on the line $\mathbb{P}^1_\ell \subset \mathbb{P}^2_\ell$, then the stabilizer of the line is a subgroup $P(U(1,\ell)\times U(2,\ell) \subset PU(3,\ell)$. To a partial spread of the set $\mathcal{H}(\ell)$ of ℓ-valued isotropic points in $\mathbb{P}^2_\ell$ corresponds a set of subgroups $P(U(1,\ell)\times U(2,\ell)$. The buildings corresponding to these groups are sub-buildings of the building of $PU(3,\ell)$. Since the lines of the spread have no isotropic points in common, the sub-buildings corresponding to these lines do not have a vertex in common. We call a covering of the building of $PU(3,\ell)$ by $SU(2,\ell)$-sub-buildings that correspond to a partial spread of the isotropic points of $\mathbb{P}^2(\ell)$ a *transversal* covering. If the partial spread is a complete spread, then the covering covers the entire building. Then it is called *complete*.

2.6 Remark. Complete transveral coverings of the $PU(3,\ell)$-building by $SU(2,\ell)$-sub-buildings exist, since spreads exist. By example 2.2, there exist for certain subgroups $G_0 \subset PU(3,\ell)$ spreads that are G_0-invariant.

2.7 Lemma. *Let h_ℓ denote a non-degenerated hermitian form on $\mathbb{P}^2_\ell$. Let $x, y \in \mathbb{P}^2_\ell$ be two distinct ℓ-valued anisotropic points. Let $z \in \mathbb{P}^2_\ell$ be the unique point that is orthogonal to both x and y w.r.t. h_ℓ. Then the following statements are equivalent:*

i) The restriction of the hermitian form h_ℓ to the line $L_{x,y} :=< x, y >$ is non-degenerated.

ii) The point $z \in \mathbb{P}^2_\ell$ that is orthogonal to both x and y is anisotropic.

iii) $h_\ell(x,x)h_\ell(y,y) - h_\ell(x,y)h_\ell(y,x) \neq 0$.

iv) The buildings belonging to the groups $PU(2,\ell)$ that stabilise x and y are disjoint.

Proof. The hermitian form h_ℓ is non-degenerated. Hence $h_\ell(z,z) \neq 0$ if the restriction of h_ℓ to the line $L_{x,y} =< x, y >$ is non-degenerated. If $h_\ell(z,z) = 0$, then $z \in L_{x,y}$ and the restriction of h_ℓ to the line $L_{x,y} =< x, y >$ is degenerated. Therefore statements (i) and (ii) of the lemma are equivalent.

Let $u \in L_{x,y}$ be the point $u := h_\ell(y,x)x - h_\ell(x,x)y$. Then $h_\ell(u,x) = h_\ell(y,x)h_\ell(x,x) - h_\ell(x,x)h_\ell(y,x) = 0$ and $h(u,u) = h_\ell(x,x) \cdot (h_\ell(x,x)h_\ell(y,y) - h_\ell(y,x)h_\ell(x,y))$. Therefore the restriction of h_ℓ to the line $L_{x,y} =< x, y >$ is degenerated if and only if $h(u,u) = 0$. Hence statements (i) and (iii) of the lemma are equivalent.

Let G_x and G_y be the stabilisers in the group $PU(3,\ell)$ of the points x and y, respectively. Then the groups G_x and G_y are isomorphic to $P(U(2,\ell) \times U(1,\ell))$. A subgroup $PU(2,\ell)$ of G_x acts on the line $x^\perp \subset \mathbb{P}^2_\ell$. Its Borel subgroups are the stabilisers of the ℓ-valued isotropic points in $x^\perp$. Hence the subgroups $PU(2,\ell)$ of G_x and G_y have a Borel group in common if and only if the point $z \perp x, y$ is isotropic.

Let $\mathbf{b}_x$ and $\mathbf{b}_y$ be the $PU(2,\ell)$-buildings belonging to x and y, respectively. The intersection $\mathbf{b}_x \bigcap \mathbf{b}_y$ is non-empty if and only if the groups

$PU(2,\ell)$ in the stabilisers of x and y have a Borel group in common. In particular, $\mathbf{b}_x \cap \mathbf{b}_y = \emptyset$ if and only if the point $z \perp x, y$ is anisotropic. $\square$

3 A forest of trees

We define (partial) coverings $\mathcal{T}$ of $\mathcal{B}$ by $PU(2,L)$-subbuildings $\mathbf{b}$ such that an edge $\mathbf{e} \in \mathcal{B}$ is contained in at most one building $\mathbf{b} \in \mathcal{T}$ and such that all hyperspecial vertices of the building are contained in the union $\bigcup_{\mathbf{b}\in\mathcal{T}} \mathbf{b}$.

These coverings $\mathcal{T}$ are the p-adic analogs of the coverings of the building of the group $PU(3,\ell)$ that correspond to (partial) spreads. A subbuilding $\mathbf{b} \subset \mathcal{B}$ corresponds to the line $\mathbb{P}^1_L \subset \mathbb{P}^2_L$ that is stabilised by the group $PU(2,L)$ determined by $\mathbf{b}$. In particular, a (partial) covering $\mathcal{T}$ defines a (partial) partition of the set of isotropic points in $\mathbb{P}^2(L)$.

We show that the automorphism group $Aut(\mathcal{T}) \subset PU(3,L)$ is a discrete subgroup. We furthermore prove that for a discrete co-compact subgroup $\Gamma \subset PU(3,L)$ there exist at most finitely many Γ-invariant coverings $\mathcal{T}$ of $\mathcal{B}$.

3.1 Remark. Let $N_{L/K} : L \longrightarrow K$ be the norm map. Since the extension $L \supset K$ is an unramified and quadratic, $N_{L/K}((L^\circ)^*) = (K^\circ)^*$ and $K^*/N_{L/K}(L^*) = \{1, \pi\}$ hold. In particular, one can rescale any point $x \in \mathbb{P}^2(L)$ such that $h(x,x) \in \{0,1,\pi\}$. The three types of points $x \in \mathbb{P}^2(L)$ correspond to three types of lines $x^\perp \cong \mathbb{P}^1_L \subset \mathbb{P}^2_L$.

If $h(x,x) = 0$, then the unitary form on $x^\perp$ is degenerated. The line $x^\perp$ contains a single L-valued isotropic point, namely the point x itself.

If $h(x,x) = \pi$, then the hermitian form on $x^\perp$ does not represent 0 over the field L. The stabiliser of the line $x^\perp$ is a compact subgroup of $PU(3,L)$.

If $h(x,x) = 1$, then the line $x^\perp$ contains infinitely many L-valued isotropic points. The stabiliser of the line $x^\perp$ in the group $PU(3,L)$ is a group $P(U(1,L) \times U(2,L))$. The group $U(1,L)$ is a finite cyclic group of order $q+1$ acting trivially on the line $x^\perp$.

Every subgroup $PU(2,L) \subset PU(3,L)$ stabilises a unique anisotropic point $x \in \mathbb{P}^2(L)$ such that $h(x,x) = 1$ after some rescaling. Hence we have a bijection between $PU(2,L)$-buildings $\mathbf{b} \subset \mathcal{B}$ and anisotropic points $x \in \mathbb{P}^2(L)$ such that $h(x,x) = 1$ after rescaling.

3.2 Definition. Let $\mathbf{b}_1, \mathbf{b}_2 \subset \mathcal{B}$ be two sub-buildings belonging to subgroups $PU(2,L) \subset PU(3,L)$. We say that the buildings $\mathbf{b}_1$ and $\mathbf{b}_2$ intersect *transversally* if they have no edge in common. The trees $\mathbf{b}_1$ and $\mathbf{b}_2$ are called *transversal* if they intersect transversally. If $\mathbf{b}_1 \cap \mathbf{b}_2 \neq \emptyset$, then the vertex $\mathbf{v} := \mathbf{b}_1 \cap \mathbf{b}_2$ is hyperspecial.

Let $\mathbf{v} \in \mathcal{B}$ be a hyperspecial vertex. Let $\mathbf{b}_1, \ldots, \mathbf{b}_s \subset \mathcal{B}$ be subbuildings belonging to subgroups $PU(2, L) \subset PU(3, L)$. We assume that the buildings intersect transversally in the hyperspecial vertex $\mathbf{v}$. Then $\bigcap_{i=1}^{s} \mathbf{b}_i = \mathbf{v}$.

The buildings $lk_{\mathbf{b}_i}(\mathbf{v})$, $i = 1, \ldots, s$ form a transversal covering of the $PU(3, \ell)$-building $lk_{\mathcal{B}}(\mathbf{v})$. Here $lk_{\mathbf{b}_i}(\mathbf{v})$ denotes the $PU(2, \ell)$-building that consists of the vertices $\mathbf{v}' \in \mathbf{b}_i$ that form an edge with the vertex $\mathbf{v}$.

Let $[M_{\mathbf{v}}]$ be the equivalence class of L°-modules belonging to the vertex $\mathbf{v}$. Each anisotropic point $\overline{x} \in \mathbb{P}(M_{\mathbf{v}} \otimes \ell)$ can be lifted to a point $x \in \mathbb{P}^2_L$. The point $x \in \mathbb{P}^2_L$ is stabilized by a unique subgroup $PU(2, L) \subset PU(3, L)$. This subgroup $PU(2, L)$ determines a subbuilding $\mathbf{b} \subset \mathcal{B}$. The edges $\mathbf{e} \in \mathbf{b}$ that contain the vertex $\mathbf{v}$ are uniquely determined by the point $\overline{x} \in \mathbb{P}(M_{\mathbf{v}} \otimes \ell)$.

Therefore each transversal covering of the building $lk_{\mathcal{B}}(\mathbf{v})$ can be lifted to characteristic zero and gives a set of $PU(2, L)$-buildings that intersect transversally at the vertex $\mathbf{v}$.

3.3 Definition. Let $\mathcal{T}$ be a set of $PU(2, L)$-sub-buildings $\mathbf{b} \subset \mathcal{B}$ that intersect transversally. If all hyperspecial vertices $\mathbf{v} \in \mathcal{B}$ are contained in the union $\bigcup\{\mathbf{b} \in \mathcal{T}\} \subset \mathcal{B}$, then we call $\mathcal{T}$ an *(almost complete) transversal covering* of the building $\mathcal{B}$. A transversal covering $\mathcal{T}$ of $\mathcal{B}$ is called *complete* if the union of the buildings $\mathbf{b} \in \mathcal{T}$ equals the entire building $\mathcal{B}$. If $\mathcal{T}$ is complete, then an edge $\mathbf{e} \in \mathcal{B}$ is contained in a unique tree $\mathbf{b} \in \mathcal{T}$.

Let $\mathbf{v}_1 \in \mathcal{B}$ be a non-hyperspecial vertex that is not contained in the union of the buildings $\mathbf{b} \in \mathcal{T}$. Since all the vertices that form an edge with $\mathbf{v}_1$ are hyperspecial, they are contained in the union $|\mathcal{T}| := \bigcup\{\mathbf{b} \in \mathcal{T}\}$. Therefore only isolated vertices are omitted from the building $\mathcal{B}$.

3.4 Example. Let us give an example of a torsion-free discrete co-compact subgroup $\Gamma \subset PU(3, L)$ that leaves invariant a complete transversal covering $\mathcal{T}$ of the building $\mathcal{B}$. Let $\mathbf{v} \in \mathcal{B}$ be a hyperspecial vertex and let $\mathcal{T}_{\mathbf{v}} := \{\mathbf{b}_i \mid i = 1, \ldots, q^2 - q + 1\}$ be a set of $PU(2, L)$-subbuildings of $\mathcal{B}$ such that for all pairs $\mathbf{b}, \mathbf{b}' \in \mathcal{T}_{\mathbf{v}}$, $\mathbf{b} \neq \mathbf{b}'$ the intersection equals $\mathbf{b} \cap \mathbf{b}' = \mathbf{v}$. For each $PU(2, L)$-building $\mathbf{b} \in \mathcal{T}_{\mathbf{v}}$ we fix a torsion-free discrete co-compact subgroup $\Gamma_{\mathbf{b}} \subset SU(2, L) \cong SL(2, K)$ that acts transitively on the two types of vertices contained in $\mathbf{b}$. Then $\mathbf{b}/\Gamma_{\mathbf{b}}$ consists of two vertices joined by $q+1$ edges. The group $\Gamma \subset PU(3, L)$ generated by the groups $\Gamma_{\mathbf{b}}$ with $\mathbf{b} \in \mathcal{T}_{\mathbf{v}}$ is a torsion-free discrete co-compact subgroup. The quotient $\mathcal{B}/\Gamma$ consists of $q^2 - q + 1$ non-hyperspecial vertices and a single hyperspecial vertex. Each non-hyperspecial vertex is joined to the hyperspecial vertex by $q + 1$ edges. The covering $\mathcal{T} := \{\gamma(\mathbf{b}) \mid \gamma \in \Gamma, \mathbf{b} \in \mathcal{T}_{\mathbf{v}}\}$ is a Γ-invariant complete transversal covering of the building $\mathcal{B}$.

In prop. 13.12 below more examples of complete transversal coverings invariant under the a discrete co-compact subgroup of $PU(3, L)$ are discussed.

3.5 Notation. For a $PU(2, L)$-subbuilding $\mathbf{b} \subset \mathcal{B}$, we denote by $H_{\mathbf{b}} \cong P(U(1, L) \times U(2, L)) \subset PU(3, L)$ the stabiliser of $\mathbf{b}$. We denote by $x_{\mathbf{b}} \in \mathbb{P}^2(L)$ the unique anisotropic point that is stabilised by the group $H_{\mathbf{b}}$.

3.6 Definition. Let $w, x, y, z \in \mathbb{P}^2(L)$ be four distinct points. We say that the points w, x, y, z are *in general position* if and only if no three of them are contained in a single line $\mathbb{P}^1_L \subset \mathbb{P}^2_L$. If four points in $\mathbb{P}^2(L)$ are in general position, then the subgroup of $PGL(3, L)$ acting on $\mathbb{P}^2_L$ that preserves the set consisting of these four points is finite.

3.7 Lemma. *Let $\mathcal{T}$ be an almost complete transversal covering of the building $\mathcal{B}$. Then the following holds:*

i) Let $\mathbf{b} \neq \mathbf{b}' \in \mathcal{T}$. Then the unitary form on the line $\langle x_{\mathbf{b}}, x_{\mathbf{b}'} \rangle$ is non-degenerated and non-compact.

ii) Let $\mathbf{b}, \mathbf{b}' \subset \mathcal{B}$, $\mathbf{b} \neq \mathbf{b}'$ be subbuildings. If $x_{\mathbf{b}'} \in x_{\mathbf{b}}^{\perp}$, then the intersection $\mathbf{b}' \cap \mathbf{b}$ consists of a single vertex.

Proof. let $x \in \mathbb{P}^2(L)$ be the unique point $x \perp \langle x_{\mathbf{b}}, x_{\mathbf{b}'} \rangle$. Since the buildings $\mathbf{b}$ and $\mathbf{b}'$ intersect transversally, $h(x, x) \neq 0$ holds. Let us assume $h(x, x) = \pi$ and derive a contradiction.

Let $f_0 \in L^3$ be a vector such that $x = [f_0]$ holds. Let $f_1, f_2 \in L^3$ be vectors such that $x_{\mathbf{b}}, x_{\mathbf{b}'} \in \langle f_1, f_2 \rangle$ holds and, moreover, the hermitian form equals $\pi y_1 \overline{y_1} + y_2 \overline{y_2}$ w.r.t. these vectors. Any L-valued point $y \in x^{\perp}$ such that $h(y, y) = 1$ has the form $y = [a f_1 + b f_2]$ with $a \in L^{\circ}$ and $b \in (L^{\circ})^*$. In particular, such points are contained in the L°-module $M := \langle f_0, f_1, f_2 \rangle$ and have a non-zero reduction w.r.t. this basis. The L°-module M corresponds to a non-hyperspecial vertex $\mathbf{v}_1 \in \mathcal{B}$. In particular, the edges $\mathbf{e} \ni \mathbf{v}_1$ are contained in the intersection $\mathbf{b} \cap \mathbf{b}'$. This cannot be and statement (i) must hold.

Let us now prove statement (ii). Let $x \in \mathbb{P}^2(L)$ be the unique point such that $x \perp x_{\mathbf{b}}, x_{\mathbf{b}'}$. Since $x_{\mathbf{b}} \perp x_{\mathbf{b}'}$, the point x is such that $h(x, x) = 1$ holds. In particular, the buildings $\mathbf{b}$ and $\mathbf{b}'$ intersect transversally.

There exist vectors $f_0, f_1, f_2 \in L^3$, such that $x = [f_0], x_{\mathbf{b}} = [f_1]$ and $x_{\mathbf{b}'} = [f_2]$. Since $h(f_i, f_i) = 1$ for $i = 0, 1, 2$, the L°-module $\langle f_0, f_1, f_2\rangle$ corresponds to a hyperspecial vertex $\mathbf{v}_0 \in \mathcal{B}$. Therefore $\mathbf{v}_0 \in \mathbf{b} \cap \mathbf{b}'$. Since the buildings intersect transversally, this is the only vertex in the intersection. $\square$

3.8 Proposition. *Let $\mathcal{T}$ be an almost complete transversal covering of $PU(2, L)$-buildings. Then the automorphism group $Aut(\mathcal{T}) \subset PU(3, L)$ is discrete.*

Proof. To prove the proposition, it is sufficient to show that for each hyperspecial vertex $\mathbf{v} \in \mathcal{B}$ the stabiliser $Aut(\mathcal{T})_{\mathbf{v}} \subset Aut(\mathcal{T})$ of the vertex is finite. To prove the finiteness of the group $Aut(\mathcal{T})_{\mathbf{v}}$, we will construct four buildings $\mathbf{b}_0, \mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3 \in \mathcal{T}$, such that $d_{\mathcal{B}}(\mathbf{v}, \mathbf{b}_i) < R$ for some finite $R \in \mathbb{R}$ and, moreover, the points $x_{\mathbf{b}_i}$, $i = 0, \ldots, 3$ are in general position. Then both the $Aut(\mathcal{T})_{\mathbf{v}}$-orbit of the set $\{x_{\mathbf{b}_i} \mid i = 0, \ldots, 3\}$ and the stabiliser in $Aut(\mathcal{T})_{\mathbf{v}}$ of this set are finite. In particular, the group $Aut(\mathcal{T})_{\mathbf{v}}$ is finite. Hence the group $Aut(\mathcal{T}) \subset PU(3, L)$ is discrete.

Let $\mathbf{v}_0 \in \mathcal{B}$ be a hyperspecial vertex. We will use induction to construct the four buildings $\mathbf{b}_0, \mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3 \in \mathcal{T}$ needed to establish the finiteness of the group $Aut(\mathcal{T})_{\mathbf{v}_0}$.

Let us assume that for some $i > 0$, we have vertices $\mathbf{v}_j$ and buildings $\mathbf{b}_j \in \mathcal{T}$ for $j = 0, \ldots, i$. Then we construct a vertex $\mathbf{v}_{i+1}$ and a building $\mathbf{b}_{i+1} \in \mathcal{T}$ such that $\mathbf{v}_{i+1} \in \mathbf{b}_{i+1}$. For $n, m \in \mathbb{N}$ we will denote by $\mathbf{b}_{m,n} \subset \mathcal{B}$, $n > m$, the subbuilding belonging to the stabiliser of the line $\langle x_{\mathbf{b}_n}, x_{\mathbf{b}_m}\rangle$.

We establish the base case $i = 0$ by choosing a building $\mathbf{b}_0 \in \mathcal{T}$ such that $\mathbf{v}_0 \in \mathbf{b}_0$.

Let us now construct a hyperspecial vertex $\mathbf{v}_i \in \mathcal{B}$ and a building $\mathbf{b}_i \in \mathcal{T}$ for $0 < i \leq 3$. Let us assume that we have constructed vertices $\mathbf{v}_j$ and buildings $\mathbf{b}_j$ for $0 \leq j < i$. We choose a vertex $\mathbf{v}' \in \mathcal{B}$ such that $d_{\mathcal{B}}(\mathbf{v}', \mathbf{v}_{i-1}) = 2$ and if $i > 1$ also $d_{\mathcal{B}}(\mathbf{v}', \mathbf{v}_{i-2}) = d_{\mathcal{B}}(\mathbf{v}_{i-1}, \mathbf{v}_{i-2}) + 2$. If $i > 1$, then we assume that $\mathbf{v}' \notin \mathbf{b}_{j,i-1}$, $j < i - 1$. We then choose a vertex $\mathbf{v}'' \in \mathcal{B}$ such that $d_{\mathcal{B}}(\mathbf{v}'', \mathbf{v}') = 2$ and $d_{\mathcal{B}}(\mathbf{v}'', \mathbf{v}_{i-1}) = 4$. If there exists a building $\mathbf{b}' \in \mathcal{T}$ such that $\mathbf{v}', \mathbf{v}_{i-1} \in \mathbf{b}'$, then we choose the vertex $\mathbf{v}'' \notin \mathbf{b}'$. We now choose a building $\mathbf{b} \in \mathcal{T}$ such that $\mathbf{v}'' \in \mathbf{b}$ and define $\mathbf{b}_i := \mathbf{b}$. If $\mathbf{v}' \in \mathbf{b}_i$, then we define $\mathbf{v}_i := \mathbf{v}'$, otherwise we define $\mathbf{v}_i := \mathbf{v}''$.

By construction $\mathbf{v}_{i-1} \notin \mathbf{b}_i$, and if $i > 1$, then $\mathbf{b}_i \cap \mathbf{b}_{j,i-1} = \emptyset$ for $0 \leq j < i - 1$. In particular, by lemma 3.7 (ii) for $i > 1$ $x_{\mathbf{b}_i} \notin \langle x_{\mathbf{b}_j}, x_{\mathbf{b}_{i-1}}\rangle$ with

$0 \le j < i-1$. Moreover, if $i > 2$, then $\mathbf{v}_{i-1} \notin \mathbf{b}_{i-3,i-2}$ and $\mathbf{b}_i \cap \mathbf{b}_{i-3,i-2} = \emptyset$. Therefore $x_{\mathbf{b}_i} \notin \langle x_{\mathbf{b}_{i-3}}, x_{\mathbf{b}_{i-2}} \rangle$.

We claim that the points $x_{\mathbf{b}_j}$, $0 \le j \le 3$ are in general position and that $d_{\mathcal{B}}(\mathbf{v}_0, \mathbf{b}_j) = d_{\mathcal{B}}(\mathbf{v}_0, \mathbf{v}_j) \le 12$, $0 \le j \le 3$ holds. The details are left to the reader. $\square$

3.9 Corollary. *Let $\mathcal{T}$ be an almost complete transversal covering of the building $\mathcal{B}$. Let $\Gamma_1 \ne \Gamma_2 \subset PU(3, L)$ be discrete co-compact subgroups. If Γ_1 and Γ_2 both preserve $\mathcal{T}$, then the groups Γ_1 and Γ_2 are commensurable.*

Proof. The group $\langle \Gamma_1, \Gamma_2 \rangle$ preserves $\mathcal{T}$. Hence the subgroup $\langle \Gamma_1, \Gamma_2 \rangle \subset PU(3, L)$ is discrete. Since $\Gamma_1, \Gamma_2 \subset PU(3, L)$ are co-compact, the intersection $\Gamma_1 \cap \Gamma_2 \subset \Gamma_1, \Gamma_2$ must be of finite index. $\square$

3.10 Proposition. *Let $\Gamma \subset PU(3, L)$ be a discrete co-compact subgroup.*

i) If $\mathcal{T}$ is a Γ-invariant transversal covering and $\mathbf{b} \in \mathcal{T}$ a subbuilding, then the subgroup $\Gamma \cap H_{\mathbf{b}} \subset H_{\mathbf{b}}$ is discrete and co-compact.

ii) There exist at most finitely many Γ-invariant almost complete transversal coverings of $\mathcal{B}$.

Proof. Let us first prove statement (i). The discreteness of $\Gamma \cap H_{\mathbf{b}} \subset H_{\mathbf{b}}$ is obvious. Let us therefore assume that the group $\Gamma \cap H_{\mathbf{b}} \subset H_{\mathbf{b}}$ is not co-compact and derive a contradiction. There exists a vertex $\mathbf{v} \in \mathbf{b}$ such that the intersection $\mathbf{b} \cap \Gamma \cdot \mathbf{v}$ consists of infinitely many $\Gamma \cap H_{\mathbf{b}}$-orbits. Let the set $\{\mathbf{v}_i \in \mathbf{b} \mid i \in I\}$ consist of representatives of the $\Gamma \cap H_{\mathbf{b}}$-orbits in the set $\mathbf{b} \cap \Gamma \cdot \mathbf{v}$. Then there exist elements $\gamma_i \in \Gamma$, $i \in I$, such that $\gamma_i(\mathbf{v}_i) = \mathbf{v}$. In particular, $\mathbf{v} \in \gamma_i(\mathbf{b}) \in \mathcal{T}$, $i \in I$ and the set $\{\gamma_i(\mathbf{b}) \mid i \in I\}$ is non-finite. This cannot be, since $\mathcal{T}$ is a transversal covering and therefore the set $\{\mathbf{b} \in \mathcal{T} \mid \mathbf{v} \in \mathbf{b}\}$ is finite. Therefore statement (i) of the proposition holds.

Let us now prove statement (ii). It is sufficient to prove the statement for groups Γ that contain no elements of finite order. To prove the statement we assume that there exist infinitely many Γ-invariant transversal coverings $\mathcal{T}$ of $\mathcal{B}$ and derive a contradiction. Let $\varphi : \mathcal{B} \longrightarrow \mathcal{B}/\Gamma$ be the quotient map. For each Γ-invariant transversal covering $\mathcal{T}$ the set $\{\varphi(\mathbf{b}) \mid \mathcal{B} \in \mathcal{T}\}$ consists of a finite number of subgraphs of $\mathcal{B}/\Gamma$. For $\mathbf{b}, \mathbf{b}' \in \mathcal{T}$ such that $\varphi(\mathbf{b}) \ne \varphi(\mathbf{b}')$, the intersection $\varphi(\mathbf{b}) \cap \varphi(\mathbf{b}')$ contains no edges. In particular, there exist only a finite number of distinct sets of subgraphs $\{\varphi(\mathbf{b}) \mid \mathbf{b} \in \mathcal{T}\}$ that can occur. Therefore at least one such set is the image of infinitely many Γ-invariant transversal coverings. Moreover, since this set is the image of infinitely many coverings $\mathcal{T}$, at least one of the graphs $\varphi(\mathbf{b})$ in this set must be the image of infinitely many different buildings $\mathbf{b}_i$, $i \in I$.

Since the group $H_{\mathbf{b}_i} \cap \Gamma \subset H_{\mathbf{b}_i}$ is non-empty, there exist elements $\gamma_i \in H_{\mathbf{b}_i} \cap \Gamma$ and vertices $\mathbf{v}_i \in \mathbf{b}$ such that $d_{\mathcal{B}}(\mathbf{v}_i, \gamma_i(\mathbf{v}_i)) = min\{d_{\mathcal{B}}(\gamma(\mathbf{v}), \mathbf{v}) \mid \gamma \in \Gamma \cap H_{\mathbf{b}_i}, \mathbf{v} \in \mathbf{b}_i\} := d_i$. We fix such an element $\gamma_i \in \Gamma \cap H_{\mathbf{b}_i}$ for each $i \in I$. The element γ_i stabilises a unique anisotropic point in $\mathbb{P}^2(L)$. Since this is the point $x_{\mathbf{b}_i}$, the element γ_i determines the building $\mathbf{b}_i$ and the group $H_{\mathbf{b}_i}$. The image $\varphi([\mathbf{v}_i, \gamma_i(\mathbf{v}_i)]) \subset \varphi(\mathbf{b})$ is a closed path that uniquely determines the building $\mathbf{b}_i$. Since there are infinitely many distinct buildings $\mathbf{b}_i$ with $\varphi(\mathbf{b}_i) = \varphi(\mathbf{b})$, there are infinitely many such closed paths contained in $\varphi(\mathbf{b})$. Therefore there must be such paths that traverse an edge $\mathbf{e} \in \varphi(\mathbf{b})$ multiple times.

For the building $\mathbf{b}_i$ corresponding to such a closed path, there exists edges $\mathbf{e}_1, \mathbf{e}_2 \in \mathbf{b}_i$ such that $\varphi(\mathbf{e}_1) = \varphi(\mathbf{e}_2) = \mathbf{e}$ and $d_{\mathcal{B}}(\mathbf{e}_1, \mathbf{e}_2) < d_i$. Therefore there exists an element $\gamma \in \Gamma - \Gamma \cap H_{\mathbf{b}_i}$ such that $\gamma(\mathbf{e}_1) = \gamma(\mathbf{e}_2)$. Hence the buildings $\gamma(\mathbf{b}_i)$ and $\mathbf{b}_i$ are distinct and have an edge in common. Since the buildings $\gamma(\mathbf{b}_i)$ and $\mathbf{b}_i$ are contained in a transversal covering, this cannot be. From this statement (ii) follows. $\square$

3.11 Definition. Let $\Gamma \subset PU(3, L)$ be a discrete co-compact subgroup. Let $\mathcal{T}_\Gamma$ be a Γ-invariant almost complete transversal covering. The covering $\mathcal{T}_\Gamma$ is called Γ-*adapted* if every edge $\mathbf{e} \in \mathcal{B}$ such that $\Gamma_{\mathbf{e}} \neq id$ is contained in a building $\mathbf{b} \in \mathcal{T}_\Gamma$.

Let $\mathcal{T}$ be a Γ-invariant transversal covering that is not Γ-adapted. Then there exists a subgroup $\Gamma' \subset \Gamma$ of finite index that does not contain any elements of finite order that preserve an edge not contained in $|\mathcal{T}| := \cup\{\mathbf{b} \in \mathcal{T}\}$. In particular, the covering $\mathcal{T}$ is Γ'-adapted.

3.12 Remark. Not all discrete co-compact subgroups $\Gamma \subset PU(3, L)$ preserve a transveral covering of $\mathcal{B}$. Indeed, if the stabiliser $\Gamma_{\mathbf{v}_0} \subset \Gamma$ of a hyperspecial vertex $\mathbf{v}_0$ does not preserve a transversal covering of the $PU(3, \ell)$-building $lk_{\mathcal{B}}(\mathbf{v}_0)$, then the group Γ does not preserve any transversal covering $\mathcal{T}$ of $\mathcal{B}$. By remark 2.4 this is the case if $\Gamma_{\mathbf{v}_0}$ contains a cyclic group C_{q^2-q+1}.

4 Arithmetic groups

We consider arithmetic discrete co-compact subgroups of $PU(3, L)$. Over a field of positive characteristic every unitary form of rank > 2 represents 0. Therefore arithmetic discrete co-compact groups only exist if $char(K) = 0$.

We prove that almost all arithmetic groups Γ preserve an almost complete transversal covering of the building $\mathcal{B}$. We give some examples of arithmetic discrete co-compact subgroups $\Gamma \subset PU(3, L)$ that admit an almost complete transversal covering of $\mathcal{B}$.

4.1. Arithmetic groups. Let $\mathcal{K}$ be a totally real Galois extension of $\mathbb{Q}$ and let $\mathcal{L}$ be a totally imaginary quadratic Galois extension of $\mathcal{K}$. In particular, $\mathcal{L}$ is a CM-field. Let $\mathfrak{p}$ be a prime ideal of $\mathcal{K}$ that is inert in the extension $\mathcal{L} \supset \mathcal{K}$.

Let h_0 be a positive definite hermitian form on $\mathcal{L}^3$. Let $m > 0$ be an integer such that the ideal $\mathfrak{p}^m$ is a principal ideal. Let $s_{\mathfrak{p}} \in \mathfrak{p}^m$ be a generator of the principal ideal $\mathfrak{p}^m$.

Let $\mathcal{O}_{\mathcal{K}}$ and $\mathcal{O}_{\mathcal{L}}$ denote the ring of integers of the field $\mathcal{K}$ and $\mathcal{L}$, respectively. Let $\Lambda \subset \mathcal{L}^3$ be an integer hermitian lattice. Let G_Λ be the algebraic group defined over $\mathcal{O}_{\mathcal{K}}$ that preserves the lattice Λ and the hermitian form h_0.

Let $\mathcal{K}_{\mathfrak{p}}$ and $\mathcal{L}_{\mathfrak{p}}$ denote the completion of $\mathcal{K}$ and $\mathcal{L}$, respectively, w.r.t. the ideal $\mathfrak{p}$

Then $G_\Lambda(\mathcal{O}_{\mathcal{L}}[1/s_{\mathfrak{p}}])$ is a discrete co-compact subgroup of the group $PU(3, \mathcal{L}_{\mathfrak{p}})$. Note that the group does not depend on the choice of the generator $s_{\mathfrak{p}}$. This is the type of arithmetic groups we consider.

4.2. The special genus. Let $Val(\mathcal{K})$ denote the set of valuations of $\mathcal{K}$. For $\nu \in Val(\mathcal{K})$ we denote by $\mathcal{K}_\nu$ the completion of $\mathcal{K}$ w.r.t. ν. We put $\mathcal{L}_\nu := \mathcal{L} \otimes \mathcal{K}_\nu$.

If the valuation is archimedean, then $\mathcal{K}_\nu \cong \mathbb{R}$ and $\mathcal{L}_\nu \cong \mathbb{C}$, since $\mathcal{L}$ is an imaginary quadratic extension of a totally real field $\mathcal{K}$. If ν corresponds to an inert or ramified prime ideal in the extension $\mathcal{L} \supset \mathcal{K}$, then $\mathcal{L}_\nu$ is equal to to the completion with respect to ν. If the prime ideal corresponding to ν splits, then $\mathcal{L}_\nu$ is equal to the product of the completions of $\mathcal{L}$ w.r.t. the two primes into which the prime ideal corresponding to ν splits.

Let V be the vector space $V := \mathcal{L}^3$ equiped with the positive hermitian form h_0. For the convenience of the reader, we recall the definition of the special genus $gen^\circ(\Lambda)$ of Λ (See [S] definition 1.7):

$gen^\circ(\Lambda) := \{\Lambda' \mid \exists(g \in U(V,h_0))\ \forall(\nu \in Val(\mathcal{K}))\ \exists(h \in SU(V \otimes \mathcal{K}_\nu, h_0))\ \Lambda \otimes \mathcal{O}_{\mathcal{K}_\nu} = g(h(\Lambda' \otimes \mathcal{O}_{\mathcal{K}_\nu}))\}$

4.3 Proposition. *Let $\mathfrak{p} \not| disc(\Lambda)$. For a vertex $\mathbf{v}$ of type 0 of the building of the group $PU(3, \mathcal{L}_\mathfrak{p})$ we denote by $M_\mathbf{v}$ the $\mathcal{O}_{\mathcal{L}_\mathfrak{p}}$-module corresponding to the vertex.*

- *i) The lattices $\Lambda_\mathbf{v} := \Lambda[\frac{1}{s_\mathfrak{p}}] \bigcap M_\mathbf{v}$ are contained in $gen^\circ(\Lambda)$ and a representative of every isomorphism class of lattices in $gen^\circ(\Lambda)$ occurs as a lattice $\Lambda_\mathbf{v}$ for some hyperspecial vertex $\mathbf{v}$ of the building..*
- *ii) The arithmetic group $G_\Lambda(\mathcal{O}_\mathcal{L}[\frac{1}{s_\mathfrak{p}}])$ has $\sharp gen^\circ(\Lambda)$ orbits on hyperspecial vertices in $\mathcal{B}$.*

Proof. In [S] the lattices $\Lambda' \in gen^\circ(\Lambda)$ are studied using the neighbourhood method. In particular, the definition of the neighbourhood of a lattice Λ (See [S] definitions 2.1 and 2.3) w.r.t. the prime ideal $\mathfrak{p} \subset \mathcal{L}$ is such that the lattices in the neighbourhood are exactly the integer lattices $\Lambda[\frac{1}{s_\mathfrak{p}}] \bigcap M_\mathbf{v}$ that correspond to the hyperspecial vertices $\mathbf{v} \in \mathcal{B}$.

The isomorphism classes of the lattices in the neighbourhood of the lattice Λ w.r.t. the prime $\mathfrak{p}$ are exactly the isomorphism classes of lattices in $gen^\circ(\Lambda)$ (See [S] theorem 2.10 and the remarks following it).

This shows that statement (i) of the proposition holds. The second statement of the proposition follows directly from statement (i). □

4.4. Minimum norm vectors. Let $N_{\mathcal{K}/\mathbb{Q}} : \mathcal{K} \longrightarrow \mathbb{Q}$ be the norm map of the field $\mathcal{K}$ into the field of rational numbers $\mathbb{Q}$. For a hermitian lattice Λ we denote the minimum norm of a non-zero vector in Λ by $min(\Lambda) := min(\{N_{\mathcal{K}/\mathbb{Q}}(h_0(x,x)) \mid x \in \Lambda, x \neq 0\})$. Let $maxmin(gen^\circ(\Lambda)) := max(\{min(\Lambda') \mid \Lambda' \in gen^\circ(\Lambda)\})$ denote the norm of the longest minimum norm vector of the lattices contained in the special genus of Λ. If $\mathcal{K} = \mathbb{Q}$, then the norm map $N_{\mathcal{K}/\mathbb{Q}}$ is the identity map.

For any finite set $X := \{[a_i] \mid a_i \in \Lambda, i = 1, \dots, s\} \subset [\Lambda']$ we define $N_{\Lambda'}(X)$ as follows: $N_{\Lambda'}(X) := \{min(\{N_{\mathcal{K}/\mathbb{Q}}(h_0(x,x)) \mid x \in \Lambda',\ [x] = [a]\}) \mid [a] \in X\}$.

4.5 Proposition. *Let $\Lambda' \in gen^\circ(\Lambda)$ be an integer lattice and let $x, y \in \Lambda'$ be two non-zero vectors such that $y \neq \lambda \cdot x$ for $\lambda \in \mathcal{L}^*$. Let us consider* *the inert prime ideals $\mathfrak{p}$ such that $h_0(x,x), h_0(y,y) \in \mathcal{O}_{\mathcal{K}} - \mathfrak{p}\mathcal{O}_{\mathcal{K}}$. Then the $PU(2, \mathcal{L}_{\mathfrak{p}})$-buildings $\mathbf{b}_x$ and $\mathbf{b}_y$, belonging to the stabilisers in $PU(3, \mathcal{L}_{\mathfrak{p}})$ of the points x and y,repectively, intersect transversally for all but finitely many choices of the prime ideal $\mathfrak{p}$.*

Proof. Let us fix vectors $x, y \in \Lambda'$ as in the statement of the proposition. Let $\mathfrak{p} \subset \mathcal{K}$ be a prime ideal that is inert in the extension $\mathcal{L} \supset \mathcal{K}$ and such that $h_0(x,x), h_0(y,y) \in \mathcal{O}_{\mathcal{K}} - \mathfrak{p}\mathcal{O}_{\mathcal{K}}$. Then the stabilisers of the vectors $x, y \in \Lambda' \subset V$ in $PU(3, \mathcal{L}_{\mathfrak{p}})$ contain groups $PU(2, \mathcal{L}_{\mathfrak{p}})$.

The $\mathcal{O}_{\mathcal{L}}$-lattice Λ' is the lattice $\Lambda' = \Lambda[\frac{1}{s_{\mathfrak{p}}}] \bigcap M_{\mathbf{v}} = \Lambda_{\mathbf{v}}$ for some hyperspecial vertex $\mathbf{v} \in \mathcal{B}$. Here $M_{\mathbf{v}}$ is the $\mathcal{L}^\circ_{\mathfrak{p}}$-module corresponding to the vertex $\mathbf{v}$. Therefore the intersection $\mathbf{b}_x \bigcap \mathbf{b}_y$ of the $PU(2, \mathcal{L}_{\mathfrak{p}})$-buildings $\mathbf{b}_x$ and $\mathbf{b}_y$ contains the vertex $\mathbf{v} \in \mathcal{B}$ such that $\Lambda_{\mathbf{v}} = \Lambda'$.

Since the hermitian form h_0 is positive definite on $V = \mathcal{L}^3$, the inequality $h_0(x,x)h_0(y,y) - h_0(x,y)h_0(y,x) \geq 0$ holds. Here equality holds if and only if $x = \lambda \cdot y$ for some $\lambda \in \mathcal{L}^*$. Therefore $h_0(x,x)h_0(y,y) - h_0(x,y)h_0(y,x) > 0$ must hold.

Hence for at most finitely many prime ideals $\mathfrak{p}$ the equality $h_0(x,x)h_0(y,y) - h_0(x,y)h_0(y,x) \equiv 0 \mod \mathfrak{p}$ holds. Using lemma 2.7 (iii), it follows that for all but finitely many choices of the inert prime ideal $\mathfrak{p}$ the buildings $\mathbf{b}_x$ and $\mathbf{b}_y$ intersect transversally. □

4.6 Lemma. *Let $\Lambda' \in gen^\circ(\Lambda)$ be an integer lattice and let $G_{\Lambda'}(\mathcal{O}_{\mathcal{L}})$ be the automorphism group of Λ'. Let $A_{\Lambda'}$ be the set $A_{\Lambda'} := \{[a] \mid a \in \Lambda',\ \exists(g \in G_{\Lambda'}(\mathcal{O}_{\mathcal{L}}) \setminus \{1\})\ g(a) = a \wedge g|_{a^\perp} = 1\}$. Then the following holds for almost all inert primes $\mathfrak{p}$:*
if $x \in \Lambda'$ such that $h(x,x) \equiv 0 \mod \mathfrak{p}$, then $\exists(g \in G_{\Lambda'}(\mathcal{O}_{\mathcal{L}}) \setminus \{1\})\ [g(x)] = [x] \Rightarrow \exists(a \in A_{\Lambda'})\ x \perp a$.

Proof. Let us consider the set $Z_{\Lambda'} := \{[x] \in [\Lambda'] \mid \exists(g \in G_{\Lambda'}(\mathcal{O}_{\mathcal{L}}))\ [g(x)] = [x] \wedge \nexists(a \in A_{\Lambda'})\ a \perp x\}$. One easily verifies that the set $Z_{\Lambda'}$ is finite. Therefore there exist only finitely many inert primes $\mathfrak{p} \subset \mathcal{L}$ such that there exists an element $x \in Z_{\Lambda'}$ with $h_0(x,x) \equiv 0 \mod \mathfrak{p}$. This proves the lemma. □

4.7 Definition. Let $v_{\mathfrak{p}}$ denote the additive valuation on $\mathcal{K}$ or $\mathcal{L}$ w.r.t. the prime ideal $\mathfrak{p}$, normalised such that $v_{\mathfrak{p}}(\mathcal{K}_{\mathfrak{p}} - \{0\}) = \mathbb{Z}$. For a vector $x \in \Lambda[\frac{1}{s_{\mathfrak{p}}}]$, $x \neq 0$ such that $v_{\mathfrak{p}}(h_0(x,x)) \in 2\mathbb{Z}$, we denote the $PU(2, \mathcal{L}_{\mathfrak{p}})$-building belonging to the stabiliser in $PU(3, \mathcal{L}_{\mathfrak{p}})$ of x by $\mathbf{b}_x$. Let $\mathcal{S} \subset \Lambda[\frac{1}{s_{\mathfrak{p}}}]$ be a set of vectors x such that $v_{\mathfrak{p}}(h_0(x,x)) \in 2\mathbb{Z}$. Then we define $\mathcal{T}_{\mathcal{S}}$ by $\mathcal{T}_{\mathcal{S}} := \{\mathbf{b}_x \mid x \in \mathcal{S}\}$.

4.8 Theorem. *Let $R \geq maxmin(gen^{\circ}(\Lambda))$ be an integer. For a prime ideal $\mathfrak{p} \subset \mathcal{K}$ that is inert in the extension $\mathcal{L} \supset \mathcal{K}$, we denote by $\mathcal{S}_{\mathfrak{p}} \subset \Lambda[\frac{1}{s_{\mathfrak{p}}}]$ the subset of non-zero vectors such that $h_0(x,x) \in \mathcal{O}_{\mathcal{K}}$ and $N_{\mathcal{K}/\mathbb{Q}}(h_0(x,x)) \leq R$. Let us consider the inert prime ideals $\mathfrak{p}$ such that $h_0(x,x) \notin \mathfrak{p}\mathcal{O}_{\mathcal{K}}$ for all $x \in \mathcal{S}_{\mathfrak{p}}$.*

i) Then for all but finitely many choices of the prime ideal $\mathfrak{p}$, the set $\mathcal{T}_{\mathcal{S}_{\mathfrak{p}}}$ is a $G_{\Lambda}(\mathcal{O}_{\mathcal{L}}[\frac{1}{s_{\mathfrak{p}}}])$-invariant almost complete transversal covering of $PU(2,\mathcal{L}_{\mathfrak{p}})$-buildings in the $PU(3,\mathcal{L}_{\mathfrak{p}})$-building.

ii) If $R \geq max(maxmin(gen^{\circ}(\Lambda)), \bigcup_{\Lambda' \in gen^{\circ}(\Lambda)} N_{\Lambda'}(A_{\Lambda'}))$, then $\mathcal{T}_{\mathcal{S}_{\mathfrak{p}}}$ is a $G_{\Lambda}(\mathcal{O}_{\mathcal{L}}[\frac{1}{s_{\mathfrak{p}}}])$-adapted almost complete transversal covering .

Proof. Let us first consider pairs of vectors $x, y \in \mathcal{S}_{\mathfrak{p}}$ that are contained in a fixed $\mathcal{O}_{\mathcal{L}}$-lattice $\Lambda' \in gen^{\circ}(\Lambda)$. Modulo units of $\mathcal{O}_{\mathcal{L}}$ there are only finitely many such vectors. By prop. 4.5 above, for all but finitely many choices of the prime ideal $\mathfrak{p}$ the stabilisers of vectors x and y with $x \neq \lambda \cdot y$, $\lambda \in \mathcal{L}^*$ give rise to $PU(2,\mathcal{L}_{\mathfrak{p}})$-buildings $\mathbf{b}_x$ and $\mathbf{b}_y$ that intersect transversally.

The special genus $gen^{\circ}(\Lambda)$ contains only finitely many isomorphism classes of $\mathcal{O}_{\mathcal{L}}$-lattices Λ'. For each isomorphism class of lattices, we only have to exclude finitely many prime ideals $\mathfrak{p}$. Hence for all but finitely many inert prime ideals $\mathfrak{p}$ all the $PU(2,\mathcal{L}_{\mathfrak{p}})$-buildings contained in $\mathcal{T}_{\mathcal{S}_{\mathfrak{p}}}$ intersect transversally.

Every hyperspecial vertex $\mathbf{v} \in \mathcal{B}$ is such that the lattice $\Lambda_{\mathbf{v}} := \Lambda[\frac{1}{s_{\mathfrak{p}}}] \bigcap M_{\mathbf{v}}$ is contained in $gen^{\circ}(\Lambda)$. Since $R \geq maxmin(gen^{\circ}(\Lambda)$, the lattice $\Lambda_{\mathbf{v}}$ contains at least one non-zero vector $x \in \mathcal{S}_{\mathfrak{p}}$. In particular, the vertex $\mathbf{v}$ is contained in the $PU(2,\mathcal{L}_{\mathfrak{p}})$-building $\mathbf{b}_x \in \mathcal{T}_{\mathcal{S}_{\mathfrak{p}}}$. Therefore $\mathcal{T}_{\mathcal{S}_{\mathfrak{p}}}$ is a $G_{\Lambda}(\mathcal{O}_{\mathcal{L}}[\frac{1}{s_{\mathfrak{p}}}])$-invariant almost complete transversal covering of $PU(2,\mathcal{L}_{\mathfrak{p}})$-buildings. This proves statement (i).

Let us now prove statement (ii). Let us fix a prime ideal $\mathfrak{p} \subset \mathcal{L}$. Let $\mathbf{v}' \in \mathcal{B}$ be a hyperspecial vertex and let $\Lambda_{\mathbf{v}} \in gen^{\circ}(\Lambda)$ be the $\mathcal{L}^{\circ}$-lattice contained in $M_{\mathbf{v}}$. It follows from lemma 4.6 that the buildings $\mathbf{b}_{[a]}$ for $[a] \in A_{\Lambda_{\mathbf{v}}}$ contain all the edges $\mathbf{e} \ni \mathbf{v}$ with non-trivial stabiliser in $G_{\Lambda}(\mathcal{O}_{\mathcal{L}}[\frac{1}{s_{\mathfrak{p}}}])$ for almost all primes. Now statement (ii) follows from statement (i). □

4.9 Example. Let $\mathcal{K} := \mathbb{Q}$ and let $\mathcal{L} := \mathbb{Q}(\sqrt{d})$ be an imaginary quadratic extension of $\mathcal{K}$ with $d \in \mathbb{Z}_{<0}$ square-free. We take $\Lambda \cong \mathcal{O}_{\mathcal{L}}^3$ in $\mathcal{L}^3$ with the standard hermitian form $h_0(x,y) = x_0\overline{y_0} + x_1\overline{y_1} + x_2\overline{y_2}$. Let p be a prime that is inert in the extension $\mathcal{L} \supset \mathcal{K}$.

The unimodular hermitian lattices Λ are well-studied for small values of d. For $d = -1, -2, -3, -11$ each lattice in the special genus $gen^\circ(\Lambda)$ contains minimum norm vectors of length 1. The number of non-isomorphic lattices in $gen^\circ(\Lambda)$ equals 1 if $d = -1, -3$ and $\sharp gen^\circ(\Lambda) = 2$ if $d = -2, -11$ (See [S] table 1 and [H] table 1).

Let $\mathcal{S}_p := \{x \in \Lambda \mid h_0(x,x) = p^{2n}, n \in \mathbb{Z}_{\geq 0}\}$. Then $\mathcal{T}_{\mathcal{S}_p}$ is an $G_\Lambda(\mathcal{O}_{\mathcal{L}}[\frac{1}{p}])$-invariant almost complete transversal covering of $PU(2, \mathcal{L}_{\mathfrak{p}})$-buildings.

In general, the transversal covering is not complete. The only exception occurs, when $p = 2$ and $d = -3$. Then the number of non-hyperspecial vertices $\mathbf{v}$ that are neighbours of a fixed hyperspecial vertex $\mathbf{v}_0$ equals $2^3+1 = 9$. Furthermore, the vertex $\mathbf{v}_0$ is contained in three $PU(2, \mathcal{L}_{\mathfrak{p}})$-buildings $\mathbf{b} \in \mathcal{S}_2$ and hence all the 9 vertices that are neighbours of $\mathbf{v}_0$ are contained in a building $\mathbf{b} \in \mathcal{T}_{\mathcal{S}}$. Since the group $G_\Lambda(\mathcal{O}_{\mathcal{L}}[\frac{1}{2}])$ acts transitively on vertices of type 0, it follows that $\mathcal{T}_{\mathcal{S}_2}$ is complete.

4.10 Remark. The positive definite lattices with class numbers one and two have been determined in [Ki] theorem 8.3.2. There are 37 distinct lattices with class number one. Hence there are 37 distinct arithmetic groups (upto commensurability) that act transitively on the hyperspecial vertices $\mathbf{v} \in \mathcal{B}$.

5 An equivariant étale covering of the p-adic upper half plane

A detailed description of a certain étale covering of the p-adic upper half plane Ω_1 is given. Instead of considering the group $SL(2, K)$ acting on $\mathbb{P}^1_K$, we consider the isomorphic group $SU(2, L)$ acting on $\mathbb{P}^1_L$ preserving a unitary form h_2. Then $\Omega_1 \subset \mathbb{P}^1_L$ is obtained by omitting the L-valued isotropic points from the projective line.

The finite covering Σ of the p-adic upper half plane $\Omega_1 := \mathbb{P}^1_L - \{x \in \mathbb{P}^1(L) \mid h_2(x,x) = 0\}$ is constructed by glueing affinoids. We give a pure affinoid covering of the rigid analytic variety Σ and define an action of the group $SU(2, L)$ on it. The covering $\Sigma \longrightarrow \Omega_1$ is $SU(2, L)$-equivariant finite étale of degree $q + 1$.

5.1. The building. Let us define the building $\mathbf{b}$ of $SU(2, L)$ (and of $PU(2, L)$) using equivalence classes of L°-modules in L^2 equiped with the unitary form h_2. All equivalence classes of L°-modules in L^2 give the building of $SL(2, L)$. The building of the group $SU(2, L)$ is now given by the equivalence classes of L°-modules that have a basis consisting of isotropic vectors. Let $e_1, e_2 \in L^2$ be two isotropic vectors, such that the unitary form is given by $h_2(x, y) = x_1\overline{y}_2 + x_2\overline{y}_1$. Let $M_0 =< e_1, e_2 >$ and let $M_1 =< e_1, \pi^{-1}e_2 >$. Then the vertices of the building of $SU(2, L)$ are the $SU(2, L)$-images of the equivalence classes $[M_0]$ and $[M_1]$. The edges of the building are give by the images of $\{[M_0], [M_1]\}$.

Each L°-module is equiped with a unitary form coming from the form h_2. On M_0 the unitary form is non-degenerated, whereas on M_1 it is degenerated.

Let $M^\vee$ be the dual module $M^\vee = \{x \in L^2 \mid \forall(y \in M)\ h_2(x, y) \in L^\circ\}$. Then $M_0^\vee = M_0$ and $M_1^\vee = \pi M_1$. The vertices of the building of $SU(2, L)$ are precisely the equivalence classes of L°-modules $[M]$ such that $[M^\vee] = [M]$ holds. Hence the $SU(2, L)$-building can be seen as the set of points fixed by an involution acting on the building of the group $SL(2, L)$.

On the building $\mathbf{b}$ an $SU(2, L)$-equivariant distance function $d_{\mathbf{b}}(-, -)$ exists. We normalise it such that $d_{\mathbf{b}}(\mathbf{v}, \mathbf{v}') = 1$ for vertices $\mathbf{v}, \mathbf{v}'$ that form an edge in the building.

5.2. The p-adic upper half plane. Let $\mathbf{b}$ denote the building of $SU(2, L)$. We briefly recall the standard pure affinoid covering of Ω_1. To the standard edge $\mathbf{e}_0 \in \mathbf{b}$ we associate the affinoid space $X_{\mathbf{e}_0}^{\Omega_1} \subset \Omega_1$ given by:
$1 \geq |\frac{x_1}{x_2}| \geq |\pi|$, $|\frac{x_1}{x_2} - c| = 1$, $|\frac{\pi x_2}{x_1} - c| = 1$, $\forall c \in (L^\circ)^*$ such that $c + \overline{c} = 0$.
To the vertices $\mathbf{v}_0$ and $\mathbf{v}_1$ of the edge $\mathbf{e}_0 \in \mathbf{b}$ we associate the open affinoid subspaces $X_{\mathbf{v}_0}^{\Omega_1}$ and $X_{\mathbf{v}_1}^{\Omega_1}$ of $X_{\mathbf{e}_0}^{\Omega_1}$ given by: $|\frac{x_1}{x_2}| = 1$ and $|\frac{\pi x_2}{x_1}| = 1$, respectively. Let $\mathcal{A}(X_{\mathbf{e}}^{\Omega_1})$ and $\mathcal{A}(X_{\mathbf{v}}^{\Omega_1})$ denote the affinoid algebras corresponding to the affinoid spaces $X_{\mathbf{e}}^{\Omega_1}$ and $X_{\mathbf{v}}^{\Omega_1}$, respectively.

One has a $SU(2, L)$-equivariant map $\psi : \Omega_1 \longrightarrow \mathbf{b}$. For $x \in X_{\mathbf{e}_0}^{\Omega_1}$ one puts $\psi(x) := (1 - v(\frac{x_1}{x_2})) \cdot \mathbf{v}_0 + v(\frac{x_1}{x_2}) \cdot \mathbf{v}_1$. Since $|\frac{g^* x_i}{x_i}| = 1$, $i = 1, 2$, for all $x \in X_{\mathbf{e}_0}^{\Omega_1}$ and all $g \in P_{\mathbf{e}_0}$, this map does not depend on the choice of the coordinates x_i, $i = 1, 2$. Let $x \in \Omega_1$ be a point. There exists an element $g \in SU(2, L)$ such that $g(x) \in X_{\mathbf{e}_0}^{\Omega_1}$. In this situation we put $\psi(x) = g^{-1}(\psi(g(x)))$. Then the function ψ is well-defined and $SU(2, L)$-equivariant.

5.3. Galois action. The analytical variety Ω_1 is defined over the field L. Therefore the Galois group $Gal(L/K)$ acts on Ω_1. To make the Galois action explicit, we embed Ω_1 into a product of two projective lines. Let $\mathbb{P}^1_L \times \mathbb{P}^1_L$ be the product of two projective lines with coordinates $(x_1,\ x_2)$ and $(z_1,\ z_2)$, respectively. On it we take a quadratic form $x_1 z_2 + x_2 z_1$. The action of the group $SU(2, L)$ is such that it preserves the quadratic form. An element $g \in SU(2, L)$ that acts on the coordinates x_i through a two by two matrix $M(g)$, acts on the coordinates z_i through the matrix $\overline{M(g)}$, obtained by replacing each coefficient $m_{i,j}$ of the matrix $M(g)$ by the Galois conjugate $\overline{m_{i,j}}$.

Let Ω_1 be contained in the projective line $\mathbb{P}^1_L$ defined by $x_1 z_2 + x_2 z_1 = 0$. The interchange of the coordinates x_i and z_i, $i = 1, 2$ is a combination of the action of the non-trivial element of the Galois group $Gal(L/K)$ and the diagonal element $diag(1, -1)$. Therefore (x_1, x_2) and $(\overline{z_1}, -\overline{z_2})$ denote the same point of Ω_1. One can also use the coordinates z_i, $i = 1, 2$ on Ω_1, to define a $SU(2, L)$-equivariant map $\psi^\vee :\ \Omega_1 \to \mathbf{b}$. Of course, $\psi^\vee((z_1, z_2)) = \psi((x_1, x_2))$, if (z_1, z_2) and (x_1, x_2) denote the same point in Ω_1.

Let $g_{\mathbf{e}_0} \in GU(2, L)$ be an element that permutes the two vertices of the building that are contained in the edge $\mathbf{e}_0$. We can choose the element $g_{\mathbf{e}_0}$ in such a way that $g^*_{\mathbf{e}_0} \frac{x_1}{x_2} = -\frac{\pi x_2}{x_1}$ holds. Without changing the action of g_{e_0} on the points in Ω_1, we can redefine the action on coordinates as being given by $(x_1, x_2) \to (\pi z_2, z_1)$ and $(z_1, z_2) \to (\pi x_2, x_1)$. The action of the element $g_{\mathbf{e}_0}$ now incorporates the action on Ω_1 of the non-trivial element of the Galois group $Gal(L/K)$.

An element $g \in GU(2, L)$ such that $v(\det(g)) \equiv 1 \mod 2\mathbb{Z}$ acts as a Galois element through permutation of the x_i and z_i coordinates on Ω_1. Indeed, one uses the fact that $g = h \cdot g_{\mathbf{e}_0}$ with $v(\det(h)) \equiv 0 \mod 2\mathbb{Z}$.

5.4. The étale covering. We define a covering of degree $q+1$ of the affinoids $X^{\Omega_1}_{\mathbf{e}}, X^{\Omega_1}_{\mathbf{v}} \subset \Omega_1$. Let $f_{\mathbf{e}_0} \in \mathcal{A}(X^{\Omega_1}_{\mathbf{e}_0})$ be the following function: $f_{\mathbf{e}_0}(x) := \frac{x_1}{x_2} \cdot \frac{1+(\frac{x_1}{x_2})^{(q-1)}}{1+(-\frac{\pi \cdot x_2}{x_1})^{(q-1)}}$. The element $g_{\mathbf{e}_0} \in GU(2, L)$ that permutes the two vertices contained in the edge $\mathbf{e}_0$ acts on $f_{\mathbf{e}_0}$ as $g^*_{\mathbf{e}_0} f_{\mathbf{e}_0} = -\pi / f_{\mathbf{e}_0}$. Let $h_{\mathbf{e}_0}$ be a $q+1$-th root of the function $-f_{\mathbf{e}_0}$ and let $h^\vee_{\mathbf{e}_0}$ be a $q+1$-th root of the function $\pi / f_{\mathbf{e}_0}$. By $\mathcal{A}_\Sigma$ we denote the affinoid algebra $\mathcal{A}_\Sigma := \mathcal{A}(X^{\Omega_1}_{\mathbf{e}_0}) < h_{\mathbf{e}_0}, \pi / h^{q+1}_{\mathbf{e}_0} >= \mathcal{A}(X^{\Omega_1}_{\mathbf{e}_0}) < h_{\mathbf{e}_0} >$. Let $X^\Sigma_{\mathbf{e}_0} := sp(\mathcal{A}_\Sigma)$ be the affinoid space belonging to the affinoid algebra $\mathcal{A}_\Sigma =: \mathcal{A}(X^\Sigma_{\mathbf{e}_0})$. The affinoid space $X^\Sigma_{\mathbf{e}_0}$ is defined over the field L.

Using the function $h^\vee_{\mathbf{e}_0}$ instead of $h_{\mathbf{e}_0}$, one can define the affinoid algebra $\mathcal{A}^\vee_\Sigma := \mathcal{A}(X^{\Omega_1}_{\mathbf{e}_0}) < h^\vee_{\mathbf{e}_0} >$. The affinoid spaces $X^\Sigma_{\mathbf{e}_0} = sp(\mathcal{A}_\Sigma)$ and $sp(\mathcal{A}^\vee_\Sigma)$

define the same covering of $X_{\mathbf{e}_0}^{\Omega_1}$, but the covering is defined differently over the field L. They become identical over the field L with a $q+1$-th root of π added.

Let $\pi^{\frac{1}{q+1}}$ denote a $q+1$-th root of π. We can extend the action of the element $g_{\mathbf{e}_0} \in GU(2, L)$ from $\mathcal{A}(X_{\mathbf{e}_0}^{\Omega_1} \otimes L[\pi^{\frac{1}{q+1}}])$ to $\mathcal{A}(X_{\mathbf{e}_0}^{\Sigma} \otimes L[\pi^{\frac{1}{q+1}}])$. Indeed, the action of the element $g_{\mathbf{e}_0} \in GU(2, L)$ that permutes the two vertices in $\mathbf{e}_0$ can be defined by $g_{\mathbf{e}_0}^* h_{\mathbf{e}_0} = h_{\mathbf{e}_0}^{\vee} := \zeta \cdot \pi^{\frac{1}{q+1}}/h_{\mathbf{e}_0}$. Here $\zeta \in L$ is a unit root such that $\zeta^{q+1} = -1$. Note, that $\mathcal{A}_\Sigma \otimes L[\pi^{\frac{1}{q+1}}] = \mathcal{A}(X_{\mathbf{e}_0}^{\Sigma} \otimes L[\pi^{\frac{1}{q+1}}]) = \mathcal{A}(X_{\mathbf{e}_0}^{\Omega_1} \otimes L[\pi^{\frac{1}{q+1}}]) < h_{\mathbf{e}_0} >= \mathcal{A}(X_{\mathbf{e}_0}^{\Omega_1} \otimes L[\pi^{\frac{1}{q+1}}]) < h_{\mathbf{e}_0}^{\vee} >= \mathcal{A}_\Sigma^{\vee} \otimes L[\pi^{\frac{1}{q+1}}]$.

5.5 Lemma. *The map $\varphi_{\mathbf{e}_0}: X_{\mathbf{e}_0}^{\Sigma} \to X_{\mathbf{e}_0}^{\Omega_1}$ induced by the inclusion $\mathcal{A}(X_{\mathbf{e}_0}^{\Omega_1}) \subset \mathcal{A}(X_{\mathbf{e}_0}^{\Sigma})$ has degree $q+1$ and is étale.*

Proof. The degree of the map $X_{\mathbf{e}_0}^{\Sigma} \to X_{\mathbf{e}_0}^{\Omega_1}$ is clear from the definition. Let us look at the points of ramification. For convenience we will work over the field extension $L[\pi^{\frac{1}{q+1}}]$. The function $f_{\mathbf{e}_0}$ has absolute value 1 outside the affinoid subspaces $X_{\mathbf{v}_0}^{\Omega_1} \otimes L[\pi^{\frac{1}{q+1}}]$ and $X_{\mathbf{v}_1}^{\Omega_1} \otimes L[\pi^{\frac{1}{q+1}}]$ of $X_{\mathbf{e}_0}^{\Omega_1} \otimes L[\pi^{\frac{1}{q+1}}]$. Moreover, the element $g_{\mathbf{e}_0} \in GU(2, L)$, that permutes the vertices $\mathbf{v}_0$ and $\mathbf{v}_1$, also permutes the ramification points of the map $X_{\mathbf{e}_0}^{\Sigma} \otimes L[\pi^{\frac{1}{q+1}}] \to X_{\mathbf{e}_0}^{\Omega_1} \otimes L[\pi^{\frac{1}{q+1}}]$. Hence it is sufficient to look at the ramification points of $X_{\mathbf{v}_0}^{\Sigma} \otimes L[\pi^{\frac{1}{q+1}}] \to X_{\mathbf{v}_0}^{\Omega_1} \otimes L[\pi^{\frac{1}{q+1}}]$. Clearly, $f_{\mathbf{e}_0}(x) = 0$ can only occur for $x \in X_{\mathbf{v}_0}^{\Omega_1} \otimes L[\pi^{\frac{1}{q+1}}]$, if $1 + (\frac{x_1}{x_2})^{q-1} = 0$. Solving this equation over the residue field, gives the isotropic points $a \in \ell^*$. Since the ℓ-valued isotropic points do not occur in the reduction of $X_{\mathbf{v}_0}^{\Omega_1} \otimes L[\pi^{\frac{1}{q+1}}]$, the map is étale. This proves the lemma. $\square$

5.6 Lemma. *Let $P_{\mathbf{e}_0} \subset SU(2, L)$ be the stabiliser of the edge $\mathbf{e}_0 \in \mathbf{b}$ and let $g \in P_{\mathbf{e}_0}$. Then the following holds:*

i) *There exists a function $C_{0,g}(x) \in \mathcal{A}(X_{\mathbf{e}_0}^{\Omega_1})$ such that $g^* f_{\mathbf{e}_0}(x) = (\frac{x_2}{g^* x_2})^{q+1} \cdot C_{0,g}(x) \cdot f_{\mathbf{e}_0}(x)$. Moreover, $C_{0,g}(x) \equiv 1$ mod π if $1 \geq |\frac{x_1}{x_2}| > |\pi|$.*

ii) *There exists a function $C_{1,g}(x) \in \mathcal{A}(X_{\mathbf{e}_0}^{\Omega_1})$ such that $g^* f_{\mathbf{e}_0}(x) = (\frac{g^* x_1}{x_1})^{q+1} \cdot C_{1,g}(x) \cdot f_{\mathbf{e}_0}(x)$. Moreover, $C_{1,g}(x) \equiv 1$ mod π if $1 > |\frac{x_1}{x_2}| \geq |\pi|$.*

Proof. We can write $f_{\mathbf{e}_0}(x) = \frac{1}{x_2^{q+1}} \cdot \frac{x_1^q x_2 + x_1 x_2^q}{1 + (-\frac{\pi \cdot x_2}{x_1})^{(q-1)}}$. Furthermore, $C_{0,g}(x) = \frac{g^* x_2^{q+1} g^* f_{\mathbf{e}_0}(x)}{x_2^{q+1} f_{\mathbf{e}_0}(x)}$ satisfies the first part of statement (i) of the lemma. It remains to show that $C_{0,g}(x) \equiv 1$ mod π for $1 \geq \frac{x_1}{x_2} > |\pi|$.

Since $|\frac{\pi x_2}{x_1}| < 1$, $C_{0,g}(x) \equiv \frac{g^*x_1 g^*x_2^q + g^*x_1^q g^*x_2}{x_1^q x_2 + x_1 x_2^q} \mod \pi$ holds. Any element $g \in P_{\mathbf{e}_0} \subset SU(2,L)$ preserves the hermitian form $h_2(x,y) = x_1\overline{y_2} + x_2\overline{y_1}$, therefore $g^*x_1 g^*x_2^q + g^*x_1^q g^*x_2 \equiv x_1^q x_2 + x_1 x_2^q \mod \pi$ for $|\frac{x_1}{x_2}| = 1$. Moreover, $\frac{g^*x_1}{x_1} \equiv \frac{g^*x_2}{x_2} \equiv 1 \mod \pi$ if $1 > |\frac{x_1}{x_2}| > |\pi|$. Hence $C_{0,g}(x) \equiv \frac{g^*x_1 g^*x_2^q + g^*x_1^q g^*x_2}{x_1^q x_2 + x_1 x_2^q} \equiv 1 \mod \pi$ for $1 \geq |\frac{x_1}{x_2}| > |\pi|$. This proves statement (i) of the lemma.

The proof of statement (ii) of the lemma is similar. One uses the equality $f_{\mathbf{e}_0}(x) = -\frac{x_1^{q+1}}{\pi} \cdot \frac{1+(\frac{x_1}{x_2})^{(q-1)}}{x_1^q(-\pi x_2) + x_1(-\pi x_2)^q}$ and the fact that the reduction of the hermitian form $g_{\mathbf{e}_0}^* h_2$ at the vertex $\mathbf{v}'$ equals $x_1^q(-\pi x_2) + x_1(-\pi x_2)^q$ modulo π.

□

5.7. The group action. Using the lemma above one can define the action of $P_{\mathbf{e}_0}$ on $\mathcal{A}(X_{\mathbf{e}_0}^{\Sigma})$. There exist functions $c_{0,g}(x)$ and $c_{1,g}(x)$ well-defined on the open admissable subsets $1 \geq |\frac{x_1}{x_2}| > |\pi|$ and $1 > |\frac{x_1}{x_2}| \geq |\pi|$ of $X_{\mathbf{e}_0}^{\Omega_1}$, respectively, such that $c_{i,g}(x)^{q+1} = C_{i,g}(x)$ and $c_{i,g}(x) \equiv 1 \mod \pi$ for $i = 1, 2$. Then the action of $g \in P_{\mathbf{e}_0}$ on $h_{\mathbf{e}_0}$ is defined as follows:

$$g^*h_{\mathbf{e}_0}(x) = \frac{x_2}{g^*x_2} \cdot c_{0,g}(x) \cdot h_{\mathbf{e}_0}(x) \text{ if } 1 \geq |\tfrac{x_1}{x_2}| > |\pi|.$$

$$g^*h_{\mathbf{e}_0}(x) = \frac{g^*x_1}{x_1} \cdot c_{1,g}(x) \cdot h_{\mathbf{e}_0}(x) \text{ if } 1 > |\tfrac{x_1}{x_2}| \geq |\pi|.$$

Since $(\frac{x_2}{g^*x_2})^{q+1} \cdot C_{0,g}(x) = (\frac{g^*x_1}{x_1})^{q+1} \cdot C_{1,g}(x)$ and $\frac{g^*x_1}{x_1} \equiv \frac{g^*x_2}{x_2} \equiv 1 \mod \pi$ for $1 > |\frac{x_1}{x_2}| > |\pi|$, these actions coincide when they are both defined. Since $g_1^*(g_2^* f_{\mathbf{e}_0}(x)) = (g_1 g_2)^* f_{\mathbf{e}_0}(x)$, it follows from the definition that $g_1^*(g_2^* h_{\mathbf{e}_0}(x)) = (g_1 g_2)^* h_{\mathbf{e}_0}(x)$.

Moreover, the first formula also defines the action of $g \in P_{\mathbf{v}_0}$ on $\mathcal{A}(X_{\mathbf{v}_0}^{\Sigma})$, since statement (i) of the lemma above still holds in this case. Similarly, the second formula defines the action of $g \in P_{\mathbf{v}_1}$ on $\mathcal{A}(X_{\mathbf{v}_1}^{\Sigma})$.

5.8 Theorem. **(a)** *The affinoid spaces $X_{\mathbf{e}}^{\Sigma}$, $X_{\mathbf{v}}^{\Sigma}$ for vertices $\mathbf{v} \in \mathbf{b}$ and edges $\mathbf{e} \in \mathbf{b}$ glue together and form a pure affinoid covering of a separated analytical space Σ.*

(b) *The map $\varphi : \Sigma \to \Omega_1$ obtained by glueing the maps $\varphi_{\mathbf{e}}$ is $SU(2,L)$-equivariant and has degree $q+1$.*

(c) *The map $\varphi : \Sigma \to \Omega_1$ is étale.*

(d) *Then the reduction of Σ is as follows:*

1. *For each vertex of the building* $\mathbf{b}$ *curves isomorphic to the plane hermitian projective curve given by* $x_0^{q+1} + x_1 x_2^q + x_1^q x_2 = 0$ *in* $\mathbb{P}^2_\ell$*. This curve is non-singular.*
2. *The components belonging to two vertices* $\mathbf{v}_1$ *and* $\mathbf{v}_2$ *of the building* $\mathbf{b}$ *intersect in a* ℓ*-valued point if and only if the two vertices form an edge* $\mathbf{e}$ *of the building.*

(e) *The reduction of* Σ *is stable over* $L(\pi^{\frac{1}{q+1}})$*.*

Proof. That the affinoids $X_{\mathbf{e}}^{\Sigma}$ for edges $\mathbf{e} \in \mathbf{b}$ glue together and form some rigid analytic space Σ is clear. The only point of concern is the separatedness of the resulting analytical space Σ. However, from the fact that we have maps $\varphi_{\mathbf{e}} : X_{\mathbf{e}}^{\Sigma} \to X_{\mathbf{e}}^{\Omega_1}$ that coincide on $X_{\mathbf{v}}^{\Sigma}$ for all edges $\mathbf{e} \ni \mathbf{v}$ the separatedness follows. This proves statement (a).

Statements (b) and (c) of the theorem are clear from the construction of the maps $\varphi_{\mathbf{e}}^{\Sigma} : X_{\mathbf{e}}^{\Sigma} \to X_{\mathbf{e}}^{\Omega_1}$ and lemma 5.5 above.

So let us now prove statement (d) of the theorem. First we consider the reduction of the affinoid $X_{\mathbf{e}}^{\Sigma}$. We only have to consider the component corresponding to the vertex $\mathbf{v}$. The other component is isomorphic to it, since the element $g_{\mathbf{e}} \in GU(2, L)$ defined above permutes the vertices in $\mathbf{e}$ and preserves $\mathcal{A}(X_{\mathbf{e}}^{\Sigma} \otimes L[\pi^{\frac{1}{q+1}}])$. The generators of the affine ℓ-algebra giving the component for the vertex $\mathbf{v}$ are $\overline{h_{\mathbf{e}}(x)}$ and $\overline{\frac{x_1}{x_2}}$ and satisfy the equation $\overline{h_{\mathbf{e}}(x)}^{q+1} = -\overline{\frac{x_1}{x_2}} - \overline{\frac{x_1}{x_2}}^q$. Furthermore, $\overline{\frac{x_1}{x_2}} \neq a$ for isotropic $a \in \ell - \{0\}$, since the ℓ-valued isotropic points are omitted.

Let us now compare this affine algebra with an open affine subset of the curve $\mathcal{C} \subset \mathbb{P}^2_\ell$ given by the equation $x_0^{q+1} + x_1 x_2^q + x_1^q x_2 = 0$. Taking $x_2 \neq 0$, we obtain the equation $\frac{x_0}{x_2}^{q+1} + \frac{x_1}{x_2} + \frac{x_1}{x_2}^q = 0$. Removing the isotropic points $\frac{x_0}{x_2} \neq a$ for $a \in \ell - \{0\}$ results in an affine subset $A \subset \mathcal{C}$ isomorphic to the component of the reduction of the affinoid space $X_{\mathbf{e}}^{\Sigma}$ belonging to the vertex $\mathbf{v} \ni \mathbf{e}$. The group $SU(2, \ell)$ acts on $\mathbb{P}^2_\ell$ fixing the point x_0. This action of $SU(2, \ell)$ preserves the curve $\mathcal{C} \subset \mathbb{P}^2_\ell$. The affine sets $g(A)$ for $g \in SU(2, \ell)$ cover the curve $\mathcal{C}$.

Let $P_{\mathbf{v}} \subset SU(2, L)$ denote the stabilizer of the vertex $\mathbf{v} \in \mathbf{b}$. Then the component of the reduction of $X_{g(\mathbf{e})}^{\Sigma} \otimes L[\pi^{\frac{1}{q+1}}]$ belonging to the vertex $\mathbf{v}$ corresponds to the affine space $\overline{g}(A)$, where $\overline{g} \in SU(2, \ell)$ denotes the reduction of $g \in P_{\mathbf{v}}$. Hence the component of the reduction of Σ belonging to the vertex $\mathbf{v}$ is indeed a curve $\mathcal{C}$ as stated in the theorem.

A calculation of the partial derivatives shows that the curve $\mathcal{C}$ is non-singular. $\square$

5.9. Embedding the affinoid $X_{\mathbf{e}}^{\Sigma}$ into a projective plane. Let us embed the affinoid space $X_{\mathbf{e}}^{\Sigma}$ into $\mathbb{P}_L^2$. Let $x_i,\ i = 0, 1, 2$ be the coordinates of $\mathbb{P}_L^2$ and let $x_0 = 0$ define a projective line $\mathbb{P}_L^1$ inside the $\mathbb{P}_L^2$. We take the usual unitary form $x_1\overline{x_2} + x_2\overline{x_1}$ on the projective line $\mathbb{P}_L^1$. It is preserved by a group $SU(2, L)$ acting on the projective line. Then $\Omega_1 \subset \mathbb{P}_L^1$ is obtained by removing the L-valued isotropic points. The affinoid $X_{\mathbf{e}_0}^{\Omega_1} \subset \Omega_1$ consists of the points $(0, x_1, x_2)$ such that $1 \geq |\frac{x_1}{x_2}| \geq |\pi|$ with the non-zero ℓ-valued isotropic points removed from the reduction of the affinoid space. An embedding of the affinoid space $X_{\mathbf{e}}^{\Sigma}$ can now be obtained by taking the points $x \in \mathbb{P}_L^2$ such that $(0, x_1, x_2) \in X_{\mathbf{e}_0}^{\Omega_1}$ and, moreover, $(\frac{x_0}{x_2})^{q+1} = -f_{\mathbf{e}_0}((x_1, x_2)) = -\frac{x_1}{x_2} \cdot \frac{1+(\frac{x_1}{x_2})^{q-1}}{1+(-\frac{\pi \cdot x_2}{x_1})^{q-1}}$. Therefore $\frac{x_0}{x_2}$ represents the function $h_{\mathbf{e}_0}((x_1, x_2))$.

Let us use a second projective plane $\mathbb{P}_L^2$ with coordinates $z_i,\ i = 1, 2, 3$ to obtain the alternative embedding $X_{\mathbf{e}}^{\Sigma}$ into $\mathbb{P}_L^2$ based on the function $h_{\mathbf{e}_0}^{\vee}$. We relate the two projective planes using the relation $x_1 z_2 + x_2 z_1 = 0$. This identifies the two projective lines $\mathbb{P}_L^1$ given by $x_0 = 0$ and by $z_0 = 0$, respectively. The action of the group $SU(2, L)$ on the $\mathbb{P}_L^1$ using the coordinates x_1, x_2 and z_1, z_2 differs by the action of a generator of the Galois group $Gal(L/K)$. Now we can describe the embedding of the affinoid space $X_{\mathbf{e}}^{\Sigma}$ using the coordinates $z_i,\ i = 0, 1, 2$. A straightforward approach would be to use equation $(\frac{z_0}{z_2})^{q+1} = \pi / f_{\mathbf{e}_0}((z_1, z_2))$ to define the embedding of the affinoid space $X_{\mathbf{e}}^{\Sigma}$, but then the action of the element $g_{\mathbf{e}_0}$ would not be defined over L. To avoid this, we use the equation $\pi \cdot (\frac{z_0}{z_2})^{q+1} = \pi / f_{\mathbf{e}_0}((z_1, z_2))$ instead. Then $X_{\mathbf{e}}^{\Sigma}$ consists of the points $z \in \mathbb{P}_L^2$ such that $(0, z_1, z_2) \in X_{\mathbf{e}_0}^{\Omega_1}$ and $(\frac{z_0}{z_2})^{q+1} = 1/f_{\mathbf{e}_0}((z_1, z_2))$.

We can now define an action of the element $g_{\mathbf{e}_0} \in GU(2, L)$ that permutes the two projective planes as follows: $g_{\mathbf{e}_0}(x) = z = (\frac{x_1 x_2}{x_0}, -x_2, \pi x_1)$ and $g_{\mathbf{e}_0}(z) = x = (\frac{z_1 z_2}{z_0}, -z_2, \pi z_1)$. This is well-defined, once the coordinate lines $x_i = 0$ and $z_i = 0,\ i = 0, 1, 2$ are removed from the two projective planes. Then $g_{\mathbf{e}_0}^2$ acts as $g_{\mathbf{e}_0}^2 = -\pi \cdot id.$ on both projective planes (with the coordinate lines removed). For a point $x \in X_{\mathbf{e}}^{\Sigma}$, the image $z = g_{\mathbf{e}_0}(x)$ satisfies: $(\frac{z_0}{z_2})^{q+1} = (\frac{x_1 x_2}{x_0}/(\pi \cdot x_1))^{q+1} = (\frac{x_2}{\pi \cdot x_0})^{q+1} = (\frac{-f_{\mathbf{e}_0}(x)}{\pi})^{q+1} = (\frac{1}{f_{\mathbf{e}_0}(g_{\mathbf{e}_0}(x))})^{q+1} = (\frac{1}{f_{\mathbf{e}_0}((z_1, z_2))})^{q+1}$. Hence $g_{\mathbf{e}_0}(x)$ is contained in the other embedding of the affinoid $X_{\mathbf{e}}^{\Sigma}$. Therefore both embeddings of the affinoid space $X_{\mathbf{e}}^{\Sigma}$ are permuted by the element $g_{\mathbf{e}_0} \in GU(2, L)$.

Since the affinoid $X_{\mathbf{e}}^{\Sigma}$ can be embedded into the projective plane and the components of the reduction of Σ consist of plane projective curves, one can embed the complete affinoid covering of Σ in the plane. However, in general such an embedding does not have a group action on it that extends to the entire projective plane.

5.10. Open admissable subspaces. For later use some open admissable subspaces of Σ are defined. For a vertex $\mathbf{v} \in \mathbf{b}$ we define the subspace $\Sigma_{\mathbf{v}} := \{x \in \Sigma \mid d_{\mathbf{b}}(\psi(\varphi(x)), \mathbf{v}) < 1\}$. For an edge $\mathbf{e} \in \mathbf{b}$ we take $\Sigma_{\mathbf{e}} := \bigcap_{\mathbf{v}\in\mathbf{e}} \Sigma_{\mathbf{v}} = \{x \in \Sigma \mid \psi(\varphi(x)) \in \mathbf{e},\ \psi(\varphi(x)) \neq \mathbf{v}$ for vertices $\mathbf{v} \in \mathbf{e}\}$.

The group action on the affinoid $X_{\mathbf{e}}^{\Sigma}$ is defined by giving an explicit group action on $X_{\mathbf{e}}^{\Sigma} \bigcap \Sigma_{\mathbf{v}}$, where $\mathbf{v} \in \mathbf{e}$ is the vertex of type $\tau(\mathbf{v}) = 0$ and on $X_{\mathbf{e}}^{\Sigma} \bigcap \Sigma_{\mathbf{v}'}$ where $\mathbf{v}' \in \mathbf{e}$ is the vertex of type $\tau(\mathbf{v}') = 1$ that coincides on the intersection $\Sigma_{\mathbf{e}} = X_{\mathbf{e}}^{\Sigma} \bigcap \Sigma_{\mathbf{v}} \bigcap \Sigma_{\mathbf{v}'}$. In the next sections we will embed $\bigcup_{\mathbf{v}\in\mathbf{b},\tau(\mathbf{v})=0} \Sigma_{\mathbf{v}}$ and $\bigcup_{\mathbf{v}\in\mathbf{b},\tau(\mathbf{v})=1} \Sigma_{\mathbf{v}}$ into projective planes such that a discrete subgroup of the group $PU(2, L)$ acts linearly on it.

5.11 Proposition. *The subsets $\Sigma_{\mathbf{v}}$ and $\Sigma_{\mathbf{e}}$ with $\mathbf{v}, \mathbf{e} \in \mathbf{b}$ are open and admissable subsets of Σ. The covering $\{\Sigma_{\mathbf{v}}, \Sigma_{\mathbf{e}} \mid \mathbf{v}, \mathbf{e} \in \mathbf{b}\}$ is an open admissable covering of Σ.*

Proof. We will only prove that $\Sigma_{\mathbf{v}}$ is open and admissable for a vertex $\mathbf{v}$ of type $\tau(\mathbf{v}) = 0$. The other cases are similar.

The space $\Sigma_{\mathbf{v}}$ equals the finite union $\Sigma_{\mathbf{v}} = \bigcup_{\mathbf{e}\ni\mathbf{v}} \Sigma_{\mathbf{v}} \cap X_{\mathbf{e}}^{\Sigma} = \bigcup_{\mathbf{e}\ni\mathbf{v}} \{x \in X_{\mathbf{e}}^{\Sigma} \mid d_{\mathbf{b}}(\psi(\varphi(x)), \mathbf{v}) < 1\}$. Without loss of generality, we may assume that $\mathbf{v}$ is the standard vertex $\mathbf{v}_0$. Let $\mathbf{e}_0 \ni \mathbf{v}_0$ be the standard edge. Then $x \in X_{\mathbf{e}_0}^{\Sigma}$ satisfies $1 \geq |\frac{x_1}{x_2}| \geq |\pi|$. The subset $\Sigma_{\mathbf{v}_0} \cap X_{\mathbf{e}_0}^{\Sigma} \subset X_{\mathbf{e}_0}^{\Sigma}$ consists of the points $x \in X_{\mathbf{e}_0}^{\Sigma}$ such that $|\frac{\pi x_2}{x_1}| < 1$ holds. Therefore by [B-G-R] §9.1.4 prop. 5. the subset $\Sigma_{\mathbf{v}_0} \cap X_{\mathbf{e}_0}^{\Sigma} \subset X_{\mathbf{e}_0}^{\Sigma}$ is open and admissable. Moreover, the finite union $\Sigma_{\mathbf{v}_0} = \bigcup_{\mathbf{e}\ni\mathbf{v}_0} \Sigma_{\mathbf{v}_0} \cap \Sigma_{\mathbf{e}}$ of open admissable subsets is again open and admissable.

The intersection $\Sigma_{\mathbf{v}} \cap \Sigma_{\mathbf{v}'} = \Sigma_{\mathbf{e}}$ if $\mathbf{v} \neq \mathbf{v}' \in \mathbf{e}$. Since the subset $\Sigma_{\mathbf{e}} \subset \Sigma_{\mathbf{v}}$ with $\mathbf{v} \in \mathbf{e}$ is open and admissable and a vertex $\mathbf{v}$ is contained in only finitely many edges $\mathbf{e}$, the covering $\{\Sigma_{\mathbf{v}}, \Sigma_{\mathbf{e}} \mid \mathbf{v}, \mathbf{e} \in \mathbf{b}\}$ is an admissable covering of Σ. $\square$

5.12 Comparison. *Relation with Drinfel'ds system of étale coverings.* In [Dr] Drinfel'd has defined a system of étale coverings of $\Omega_{1,K} := \mathbb{P}^1_K - \mathbb{P}^1(K)$. The first level $\Sigma_K^{(q-1)}$ of this system of coverings consists of $q - 1$ connected components that are all isomorphic (See [Te] and [P] theorem 8.4). The reduction of a connected component of this first level consists of hermitian curves. Over the ring $\mathbb{Z}_p^{nr}$ the following equation for $\Sigma_K^{(q-1)}$ around a vertex in affine coordinates is derived : $Y_0^{p^2-1} = p(z_0 - z_0^p)^{p-1}$ (See [Te] cor. 6).

The constant factor p matters for a definition over K° and is irrelevant for our construction. In a semistable model the constant factor is absorbed in the coordinates. In [P] theorem 8.4 such a semistable model of $\Sigma_K^{(q-1)}$ over the field $L((-\pi)^{\frac{1}{q^2-1}})$, $\pi = p$ is described. We will now describe how a connected component of this semistable model is related to our space Σ.

Let us first construct a curve Σ_K over the field K, such that $\Sigma_K \otimes L \cong \Sigma$. To define an affinoid $X_{\mathbf{e}_0}^{\Sigma_K}$ for the edge $\mathbf{e}_0 \in \mathbf{b}$, one considers $\widetilde{f}_{\mathbf{e}_0}(x) := \frac{x_1}{x_2} \cdot \frac{1-(\frac{x_1}{x_2})^{(q-1)}}{1-(-\frac{\pi \cdot x_2}{x_1})^{(q-1)}}$. Let $\widetilde{h}_{\mathbf{e}_0}(x)^{q+1} = -\widetilde{f}_{\mathbf{e}_0}(x)$. Here $\frac{x_1}{x_2} \in \Omega_{1,K}$, $1 \geq |\frac{x_1}{x_2}| \geq |\pi|$. Then the affinoids $X_{\mathbf{e}}^{\Sigma_K}$ for the edges $\mathbf{e} \in \mathbf{b}$ glue together into a rigid analytic variety Σ_K that is an étale covering of $\Omega_{1,K}$ of degree $q+1$. The methods used above to define the group action on Σ and describe the reduction of Σ are also applicable in this case.

One can construct the isomorphism $\Sigma_K \otimes L \cong \Sigma$ explicitly. Let $\varphi_\xi : \Omega_{1,K} \otimes L \to \Omega_1$ be the isomorphism given by $\varphi_\xi((x_1, x_2)) = (x_1, \xi x_2)$, with $\xi \in L^\circ$, $\xi^{q-1} = -1$. Then $\varphi_{\xi *} f_{\mathbf{e}_0} = \xi \widetilde{f}_{\mathbf{e}_0}$. Let $\beta \in L^\circ$ be such that $\beta^{q+1} = \xi$. Then we define $\varphi_{\xi *} h_{\mathbf{e}_0} = \beta \widetilde{h}_{\mathbf{e}_0}$ and obtain an isomorphism between the affinoids $X_{\mathbf{e}_0}^{\Sigma}$ and $X_{\mathbf{e}_0}^{\Sigma_K} \otimes L$ that is defined over the unramified extension $L' \supset K$ of degree four. This isomorphism only depends on the choice of ξ. It does not depend on the value of β choosen.

The reduction at a vertex is a curve $\mathcal{H}_{-1}$ defined by the equation $x_0^{q+1} + x_1 x_2^q - x_2 x_1^q = 0$. In this case $(\frac{\pi^{\frac{1}{q+1}}}{\beta \widetilde{h}_{\mathbf{e}_0}})^{q+1} = \frac{\pi}{-\xi \widetilde{f}_{\mathbf{e}_0}(x)}$. This gives rise to the same hermitian curve $\mathcal{H}_{-1}$ as reduction at the other type of vertices.

The above gives an isomorphism over some extension $L' \supset L$. The next step is to enlarge the field and take different generators of the affinoid algebras and use these to construct an isomorphism.

Let $\zeta_c \in L^\circ$ be a unit root such that its reduction equals $c \in \ell$. Let $L'' \supset L$ be an unramified extension that contains an element β_c such that $\beta_c^{q+1} = \zeta_c$. Then we define $(e_\omega(x))^{q+1} = -\frac{1}{\zeta_c} \xi \widetilde{f}_{\mathbf{e}_0}(x)$ and $e'_\omega(x) = \xi \cdot \frac{\pi^{\frac{1}{q+1}}}{e_\omega(x)}$. In particular, $e_\omega(x) = \beta_c \beta \widetilde{h}_{\mathbf{e}_0}$, $(e'_\omega(x))^{q+1} = \xi^{q+1} \frac{\pi}{\xi \frac{1}{\zeta_c} \cdot \widetilde{f}_{\mathbf{e}_0}(x)} = -\xi^2 \frac{\pi}{\xi \frac{1}{\zeta_c} \cdot \widetilde{f}_{\mathbf{e}_0}(x)} = -\zeta_c \xi \frac{\pi}{\widetilde{f}_{\mathbf{e}_0}(x)}$ and $e'_\omega(x) = \beta_c^{-1} \beta^q \frac{\pi^{\frac{1}{q+1}}}{\widetilde{h}_{\mathbf{e}_0}}$. (Need to take β^q instead of β in $e'_\omega(x)$???)

The isomorphism $\varphi_{\xi*}$ is now given by:
$(\frac{x_1}{x_2}, \beta_c \cdot h_{\mathbf{e}_0}(x), \xi\beta_c^{-1} \cdot \frac{\pi^{\frac{1}{q+1}}}{h_{\mathbf{e}_0(x)}}) \longrightarrow (\xi\frac{x_1}{x_2}, e_\omega(x), e'_\omega(x))$. On both sides of the arrows there are generators of the affinoid algebra over the field $L''(\pi^{\frac{1}{q+1}})$. These generators can be used to define the analytic variety over the field L.

By construction the group $SL(2,K)$ acts on these varieties and each variety is after a base change isomorphic to $\Sigma \otimes L''(\pi^{\frac{1}{q+1}})$. It seems the group $GL(2,K)$ in general does not act on these varieties. If $K = \mathbb{Q}_p$, then there seem to be only $p-1$ varieties on which the group $GL(2,\mathbb{Q}_p)$ acts (See [P] lemma 7.4 and proposition 7.5). These $p-1$ varieties correspond to the connected components of $\Sigma_K^{(q-1)}$.

6 Equivariant embeddings into the projective plane

Let $\Gamma_{\mathbf{b}} \subset P(U(1,L) \times U(2,L))$ be a discrete co-compact subgroup. Let $\mathbb{P}$ and $\mathbb{P}^\vee$ be two projective planes $\mathbb{P}^2_L$ on which the group $P(U(1,L) \times U(2,L))$ acts linearly such that it fixes a single point. The action of $P(U(1,L) \times U(2,L))$ on $\mathbb{P}$ and $\mathbb{P}^\vee$ differs by conjugation with the non-trivial element of the Galois group $Gal(L/K)$.

We embed the open admissable subspaces $\Sigma_{\mathbf{v}} \subset \Sigma$, $\mathbf{v} \in \mathbf{b}$ with $\tau(\mathbf{v}) = 0$ into $\mathbb{P}$ and the subspaces $\Sigma_{\mathbf{v}}$ with $\tau(\mathbf{v}) = 1$ into $\mathbb{P}^\vee$. These embeddings are defined using infinite $\Gamma_{\mathbf{b}}$-invariant sums that converge on Ω_1. The embeddings therefore explicitly use the fact that Σ is a covering of Ω_1. Moreover, the embeddings are $\Gamma_{\mathbf{b}}$-equivariant. The spaces $\Sigma_{\mathbf{v}}$, $\mathbf{v} \in \mathbf{b}$ are glued along the spaces $\Sigma_{\mathbf{e}}$ for edges $\mathbf{e} \in \mathbf{b}$ to obtain the analytic space Σ.

6.1 An equivariant embedding

6.1. Infinite sums on Ω_1. Let $\Gamma_{\mathbf{b}} \subset P(U(1,L) \times U(2,L))$ be a discrete co-compact subgroup. Let the group $\Gamma_{\mathbf{b}}$ act on Ω_1 through the projection onto the group $U(2,L)$. We define the $\Gamma_{\mathbf{b}}$-invariant infinite sums converging on Ω_1 that can be used to obtain a $\Gamma_{\mathbf{b}}$-equivariant embedding of $\bigcup_{\mathbf{v}\in\mathbf{b},\tau(\mathbf{v})=0} \Sigma_{\mathbf{v}}$ $(\bigcup_{\mathbf{v}\in\mathbf{b},\tau(\mathbf{v})=1} \Sigma_{\mathbf{v}})$ into $\mathbb{P}^2_L$.

Let $\mathbf{v}_0 \in \mathbf{b}$ be the vertex corresponding to the equivalence class $[M_{\mathbf{v}_0}]$, $M_{\mathbf{v}_0} =< e_0, e_1, e_2 >$ and let $\mathbf{v}_1 \in \mathbf{b}$ be the vertex corresponding to the equivalence class $[M_{\mathbf{v}_1}]$, $M_{\mathbf{v}_1} =< e_0, e_1, \pi^{-1}e_2 >$. We associate to the vertices $\mathbf{v}_0, \mathbf{v}_1 \in \mathbf{b}$ polynomials $A_{\mathbf{v}_0}(x)$ and $A_{\mathbf{v}_1}(z)$ of degree $q+1$. For points $x, z \in \Omega_1$ one defines $A_{\mathbf{v}_0}(x) := x_1 x_2^q + x_2 x_1^q$ and $A_{\mathbf{v}_1}(z) := z_1(\pi z_2)^q + (\pi z_2) z_1^q$. The action of the group $U(2, L)$ on the coordinates x_i and z_i, $i = 1, 2$, differs by conjugation with the non-trivial element of the Galois group $Gal(L/K)$.

We associate to each vertex $\mathbf{v} \in \mathbf{b}$ a homogeneous polynomial $a_{\mathbf{v},\mathbf{b}}(x)$ or $a_{\mathbf{v},\mathbf{b}}(z)$ of degree $q+1$ such that the following four conditions hold:

i) If $\tau(\mathbf{v}) = 0$, then $a_{\mathbf{v},\mathbf{b}}(x) \equiv g^* A_{\mathbf{v}_0}(x) \mod \pi$ for all $g \in U(2, L)$ such that $g(\mathbf{v}_0) = \mathbf{v}$.

ii) If $\tau(\mathbf{v}) = 1$, then $a_{\mathbf{v},\mathbf{b}}(z) \equiv g^* A_{\mathbf{v}_1}(z) \mod \pi$ for all $g \in U(2, L)$ such that $g(\mathbf{v}_1) = \mathbf{v}$.

iii) If $\tau(\mathbf{v}) = 0$, $a_{\gamma(\mathbf{v}),\mathbf{b}}(x) = \gamma^* a_{\mathbf{v},\mathbf{b}}(x)$ for all $\gamma \in \Gamma_{\mathbf{b}}$.

iv) If $\tau(\mathbf{v}) = 1$, $a_{\gamma(\mathbf{v}),\mathbf{b}}(z) = \gamma^* a_{\mathbf{v},\mathbf{b}}(z)$ for all $\gamma \in \Gamma_{\mathbf{b}}$.

Let $F_{\mathbf{b}}(x) := (\sum_{\mathbf{v} \in \mathbf{b}, \tau(\mathbf{v})=0} a_{\mathbf{v},\mathbf{b}}(x)^{-1})^{-1}$ and let $F_{\mathbf{b}}^{\vee}(z) := (\sum_{\mathbf{v} \in \mathbf{b}, \tau(\mathbf{v})=1} a_{\mathbf{v},\mathbf{b}}(z)^{-1})^{-1}$. Below we show that these sums are well-defined for points $x, z \in \Omega_1$.

6.2 Lemma. *Let $x \in \Omega_1$ be a point such that $\psi(x) \in A \subset \mathbf{b}$. Then $v(\frac{g^* x_1 g^* x_2}{x_1 x_2}(x)) = -d_{\mathbf{b}}(gA, \psi(x))$.*

Proof. If $\psi(x) \in gA$, then $d_{\mathbf{b}}(gA, \psi(x)) = 0$ and we may assume that $g \in P_{\psi(x)}$. Then $|\frac{g^* x_i}{x_i}| = 1$ for $i = 1, 2$ and the lemma holds.

So let us now consider the case where $\psi(x) \notin gA$. Let $\mathbf{v} \in gA$ be the vertex closest to $\psi(x)$. Let $\mathbf{e} \in A \subset \mathbf{b}$ be the edge containing $\psi(x)$. Without loss of generality, we may assume that $\mathbf{e}$ is our standard edge. Since $\forall (h \in P_{\mathbf{e}}) \ |\frac{h^* x_i}{x_i}| = 1$, $i = 1, 2$, the absolute value of the coordinates x_1, x_2 does not depend on the choice of the apartment $A \ni \psi(x)$. Therefore we may assume that the vertex $\mathbf{v}$ is contained in A.

Let us first consider the case where the vertex $\mathbf{v}$ is of type $\tau(\mathbf{v}) = 0$. Let $[M_{\mathbf{v}}]$ be a the equivalence class of L°-modules corresponding to the vertex $\mathbf{v}$. We may assume that $M_{\mathbf{v}} =< e_1, e_2 >$ and after replacing the the coordinates x_i by suitable translates by an element of the maximal split torus belonging to the apartment A, we may assume that the coordinates x_i, $i = 1, 2$, correspond to the basis e_i of $\mathbb{P}^1_L$.

We normalise the coordinates of the point x such that $|x_i| \leq 1$ and one of the coordinates has absolute value equal to 1. Using this normalisation the equality $v(x_1 x_2) = d_{\mathbf{b}}(\mathbf{v}, \psi(x))$ holds.

Since the apartment $g(A)$ is such that the vertex $\mathbf{v} \in g(A)$ is closest to the point $\psi(x) \in A$, $|g^* x_i| = 1$ holds for $i = 1, 2$. Therefore $-v(\frac{g^* x_1 g^* x_2}{x_1 x_2}) = v(x_1 x_2) = d_{\mathbf{b}}(\mathbf{v}, \psi(x))$ for elements $g \in P_{\mathbf{v}}$ such that $g(A) \bigcap A = \{\mathbf{v}\}$.

This proves the lemma if the vertex $\mathbf{v}$ is of type $\tau(\mathbf{v}) = 0$. The proof in case the vertex $\mathbf{v}$ is of type $\tau(\mathbf{v}) = 1$ is similar. $\square$

6.3 Lemma. *Let $x \in X_{\mathbf{e}} \subset \Omega_1$ be a point and let $\mathbf{v} \in \mathbf{b}$ be a vertex. Let $A \subset \mathbf{b}$ be an apartment containing $\psi(x)$ and let x_1, x_2 be the associated coordinates. Then:*

i) If $\tau(\mathbf{v}) = 0$ and $d_{\mathbf{b}}(\mathbf{v}, \psi(x)) \leq 1$, then $v(\frac{a_{\mathbf{v},\mathbf{b}}(x)}{(x_1 x_2)^{(q+1)/2}}) = -\frac{q-1}{2} \cdot d_{\mathbf{b}}(\mathbf{v}, \psi(x))$.

ii) If $\tau(\mathbf{v}) = 0$ and $d_{\mathbf{b}}(\mathbf{v}, \psi(x)) > 1$, then $-\frac{q-1}{2} \cdot d_{\mathbf{b}}(\mathbf{v}, \psi(x)) \geq v(\frac{a_{\mathbf{v},\mathbf{b}}(x)}{(x_1 x_2)^{(q+1)/2}}) \geq -\frac{q+1}{2} \cdot d_{\mathbf{b}}(\mathbf{v}, \psi(x)) + 1$.

iii) If $\tau(\mathbf{v}) = 1$ and $d_{\mathbf{b}}(\mathbf{v}, \psi^\vee(z)) \leq 1$, then $v(\frac{a_{\mathbf{v},\mathbf{b}}(z)}{(\pi z_1 z_2)^{(q+1)/2}}) = -\frac{q-1}{2} \cdot d_{\mathbf{b}}(\mathbf{v}, \psi^\vee(z))$.

iv) If $\tau(\mathbf{v}) = 1$ and $d_{\mathbf{b}}(\mathbf{v}, \psi^\vee(z)) > 1$, then $-\frac{q-1}{2} \cdot d_{\mathbf{b}}(\mathbf{v}, \psi^\vee(z)) \geq v(\frac{a_{\mathbf{v},\mathbf{b}}(z)}{(\pi z_1 z_2)^{(q+1)/2}}) \geq -\frac{q+1}{2} \cdot d_{\mathbf{b}}(\mathbf{v}, \psi^\vee(z)) + 1$.

Proof. Let $\mathbf{e} \in A \subset \mathbf{b}$ be the edge containing $\psi(x)$. Without loss of generality, we may assume that $\mathbf{e}$ is our standard edge. Since $\forall (g \in P_{\mathbf{e}})\ |\frac{g^* x_i}{x_i}| = 1$, $i = 1, 2$, the absolute value of the coordinates x_1, x_2 does not depend on the choice of the apartment $A \ni \psi(x)$. Therefore we may assume that the vertex $\mathbf{v}$ is contained in A.

Let us first consider the case where the vertex $\mathbf{v}$ is of type $\tau(\mathbf{v}) = 0$. Since the characteristic of the residue field ℓ of L does not equal 2, the homogeneous polynomial $a_{\mathbf{v},\mathbf{b}}(x)$ of degree $q + 1$ is such that $a_{\mathbf{v},\mathbf{b}}(x) \equiv (g_0^* x_1 g_0^* x_2) \cdot (g_1^* x_1 g_1^* x_2) \cdots (g_{(q-1)/2}^* x_1 g_{(q-1)/2}^* x_2) \mod \pi$ for suitably choosen elements $g_i \in P_{\mathbf{v}},\ i = 0, \ldots, (q-1)/2$. One can choose the elements g_i such that $g_i(A) \bigcap A = \{\mathbf{v}\}$ for $i = 1, \ldots, (q-1)/2$ and $g_0(A) \bigcap A$ contains the two edges $\mathbf{e} \ni \mathbf{v}$ that are contained in A.

By lemma 6.2 above $-v(\frac{g^* x_1 g^* x_2}{x_1 x_2}) = d_{\mathbf{b}}(\mathbf{v}, \psi(x))$ for $g \in P_{\mathbf{v}}$ such that $g(A) \bigcap A = \{\mathbf{v}\}$. If $\psi(x) \in g_0(A)$, then $-v(\frac{g_0^* x_1 g_0^* x_2}{x_1 x_2}) = 0$. Therefore $v(\frac{a_{\mathbf{v},\mathbf{b}}(x)}{(x_1 x_2)^{(q+1)/2}}) = -\frac{q-1}{2} \cdot d_{\mathbf{b}}(\mathbf{v}, \psi(x))$, if $\psi(x) \in g_0(A)$. In particular, this holds if $d_{\mathbf{b}}(\mathbf{v}, \psi(x)) \leq 1$. This proves statement (i) of the lemma.

Statement (ii) of the lemma follows from the fact that $0 \leq d_{\mathbf{b}}(g_0 A, \psi(x)) \leq d_{\mathbf{b}}(\mathbf{v}, \psi(x)) - 1$, since the apartment $g_0 A$ contains the two edges $\mathbf{e} \ni \mathbf{v}$ that are contained in $A \ni \psi(x)$.

The proof of the statements (iii) and (iv) for the case $\tau(\mathbf{v}) = 1$ is similar. □

6.4 Lemma. *The following statements hold:*

i) The sums $\sum_{\mathbf{v}\in\mathbf{b},\tau(\mathbf{v})=0} a_{\mathbf{v},\mathbf{b}}(x)^{-1}$ *and* $\sum_{\mathbf{v}\in\mathbf{b},\tau(\mathbf{v})=1} a_{\mathbf{v},\mathbf{b}}(z)^{-1}$ *converge on* Ω_1.

ii) The sum $\sum_{\mathbf{v}\in\mathbf{b},\tau(\mathbf{v})=0} a_{\mathbf{v},\mathbf{b}}(x)^{-1}$ *has only zeroes at points* $x \in \Omega_1$ *such that* $\psi(x)$ *is a vertex* $\mathbf{v}$ *of type* $\tau(\mathbf{v}) = 1$.

iii) The sum $\sum_{\mathbf{v}\in\mathbf{b},\tau(\mathbf{v})=1} a_{\mathbf{v},\mathbf{b}}(z)^{-1}$ *has only zeroes at points* $z \in \Omega_1$ *such that* $\psi^{\vee}(z)$ *is a vertex* $\mathbf{v}$ *of type* $\tau(\mathbf{v}) = 0$.

Proof. From lemma 6.3 it follows that $a_{\mathbf{v},\mathbf{b}}(x)^{-1} \longrightarrow 0$, if $d_{\mathbf{b}}(\mathbf{v}, \psi(x)) \longrightarrow \infty$. Since there are only finitely many vertices $\mathbf{v} \in \mathbf{b}$, such that $d_{\mathbf{b}}(\mathbf{v}, \psi(x)) \leq R$ for any finite $R \in \mathbb{R}$, it follows that the first sum converges on all of Ω_1. The argument for the second sum is similar. This proves statement (i) of the lemma.

A zero of the sums can only occur, if more than one polynomial $a_{\mathbf{v},\mathbf{b}}(x)$ obtains the minimal absolute value of such polynomials on x. Hence more than one vertex $\mathbf{v}$ should obtain the minimum distance to the point $\psi(x) \in \mathbf{b}$. This can only occur if $\psi(x)$ is a vertex such that the type $\tau(\mathbf{v})$ is distinct from the type of the vertices used to define the sums. This proves the second statement of the lemma.

The third statement of the lemma is proved analogously. This concludes the proof of the lemma. □

6.5. The spaces $Y^s_{\mathbf{b}}$ and $Y^{s\vee}_{\mathbf{b}}$. Let us take a projective space $\mathbb{P}^5_L$ with homogeneous coordinates $(x_0, x_1, x_2, z_0, z_1, z_2)$ with a quadratic form $x_0 z_0 + x_1 z_2 + x_2 z_1$ on it. In $\mathbb{P}^5_L$ one has two projective planes $\mathbb{P}^2_L$, one defined by $z_0 = z_1 = z_2 = 0$ and one defined by $x_0 = x_1 = x_2 = 0$. Then (x_0, x_1, x_2) and (z_0, z_1, z_2) are the homogeneous coordinates on these two distinct projective planes $\mathbb{P}^2_L$. Let us denote these projective planes by $\mathbb{P}$ and $\mathbb{P}^{\vee}$,respectively.

Let the unitary group $U(1, L) \times U(2, L)$ act linearly on $\mathbb{P}^5_L$ preserving the quadratic form, the subspaces $\mathbb{P}$ and $\mathbb{P}^{\vee}$ and the points x_0 and z_0.

On $\mathbb{P}$ and $\mathbb{P}^\vee$ the group $U(1,L)\times U(2,L)$ acts linearly fixing the point x_0 and z_0, respectively. The group $U(2,L)$ acts linearly on the projective line $\mathbb{P}^1_L$ given by $x_0=0$ and $z_0=0$ in $\mathbb{P}$ and $\mathbb{P}^\vee$, respectively. The action of $U(1,L)\times U(2,L)$ on the projective planes $\mathbb{P}$ and $\mathbb{P}^\vee$ differs by conjugation with the non-trivial element of the Galois group $Gal(L/K)$.

Let us define hermitian forms h and $h^\vee$ preserved by $U(1,L)\times U(2,L)$ on $\mathbb{P}$ and $\mathbb{P}^\vee$, respectively. Let $h(x,y)=x_1\overline{y}_2+x_2\overline{y}_1+x_0\overline{y}_0$ for $x,y\in\mathbb{P}$ and let $h^\vee(z,u)=z_1\overline{u}_2+z_2\overline{u}_1+z_0\overline{u}_0$ for $z,u\in\mathbb{P}^\vee$, where the points y and u are L-valued.

Let $Y^s_{\mathbf{b}}\subset\mathbb{P}$ denote the space $Y^s_{\mathbf{b}}:=\{x\in\mathbb{P}\mid(x_1,x_2)\in\Omega_1\}$ and let $Y^{s\vee}_{\mathbf{b}}:=\{z\in\mathbb{P}^\vee\mid(z_1,z_2)\in\Omega_1\}$. The spaces $Y^s_{\mathbf{b}}$ and $Y^{s\vee}_{\mathbf{b}}$ can be described using the hermitan forms h and $h^\vee$, respectively. Indeed, $Y^s_{\mathbf{b}}=\{x\in\mathbb{P}\cong\mathbb{P}^2_L\mid\forall(y=(0,y_1,y_2)\in\mathbb{P}^2(L)$ such that $h(y,y)=0)\ h(x,y)\neq0\}$. A similar description holds for $Y^{s\vee}_{\mathbf{b}}$.

The maps $\psi,\ \psi^\vee:\Omega_1\longrightarrow\mathbf{b}$ can be extended to $Y^s_{\mathbf{b}}$ and $Y^{s\vee}_{\mathbf{b}}$, respectively. We will denote these maps by $\psi_{\mathbf{b}}$ and $\psi^\vee_{\mathbf{b}}$. Then $\psi_{\mathbf{b}}:Y^s_{\mathbf{b}}\longrightarrow\mathbf{b}$ denotes the map defined by $\psi_{\mathbf{b}}(x):=\psi((x_1,x_2))$ and $\psi^\vee_{\mathbf{b}}:Y^{s\vee}_{\mathbf{b}}\longrightarrow\mathbf{b}$ denotes the map defined by $\psi^\vee_{\mathbf{b}}(z):=\psi^\vee((z_1,z_2))$.

6.6. Analytical subspaces of $Y^s_{\mathbf{b}}$ and $Y^{s\vee}_{\mathbf{b}}$ for vertices and edges in b. For each vertex $\mathbf{v}\in\mathbf{b}$ we define an analytical space $\Sigma^\sharp_{\mathbf{v}}$. The definition of the space depends on the type $\tau(\mathbf{v})$ of the vertex. If $\tau(\mathbf{v})=0$, then we define $\Sigma^\sharp_{\mathbf{v}}$ by $\Sigma^\sharp_{\mathbf{v}}:=\{x\in Y^s_{\mathbf{b}}\mid d_{\mathbf{b}}(\psi_{\mathbf{b}}(x),\mathbf{v})<1,\ x_0^{q+1}+F_{\mathbf{b}}((x_1,x_2))=0\}$. For a vertex $\mathbf{v}$ of type $\tau(\mathbf{v})=1$, we define $\Sigma^\sharp_{\mathbf{v}}$ by $\Sigma^\sharp_{\mathbf{v}}:=\{z\in Y^{s\vee}_{\mathbf{b}}\mid d_{\mathbf{b}}(\psi^\vee_{\mathbf{b}}(z),\mathbf{v})<1,\ -\pi\cdot z_0^{q+1}+F^\vee_{\mathbf{b}}((z_1,z_2))=0\}$.

For an edge $\mathbf{e}\in\mathbf{b}$ we define two spaces $\Sigma^\sharp_{\mathbf{e}}$ and $\Sigma^{\sharp\vee}_{\mathbf{e}}$. Let $\Sigma^\sharp_{\mathbf{e}}:=\{x\in\Sigma^\sharp_{\mathbf{v}}\mid\psi_{\mathbf{b}}(x)\in\mathbf{e},\ \psi_{\mathbf{b}}(x)\neq\mathbf{v}\}\subset Y^s_{\mathbf{b}}$, where $\mathbf{v}\in\mathbf{e}$ is the vertex of type $\tau(\mathbf{v})=0$. Let $\Sigma^{\sharp\vee}_{\mathbf{e}}:=\{z\in\Sigma^\sharp_{\mathbf{v}'}\mid\psi^\vee_{\mathbf{b}}(z)\in\mathbf{e},\ \psi^\vee_{\mathbf{b}}(z)\neq\mathbf{v}'\}\subset Y^{s\vee}_{\mathbf{b}}$, where $\mathbf{v}'\in\mathbf{e}$ is the vertex of type $\tau(\mathbf{v}')=1$.

6.7 Proposition. *Let $\mathbf{v}\in\mathbf{b}$ be a vertex. Then $\Sigma^\sharp_{\mathbf{v}}\cong\Sigma_{\mathbf{v}}$.*

Proof. It is sufficient to prove the proposition for the vertices $\mathbf{v}_0$ and $\mathbf{v}_1$ of type $\tau(\mathbf{v}_i)=i$, $i=0,1$ that are contained in the standard edge $\mathbf{e}_0\in\mathbf{b}$. We have to show that $\Sigma^\sharp_{\mathbf{v}_0}\cong\Sigma_{\mathbf{v}_0}$ and that $\Sigma^\sharp_{\mathbf{v}_1}\cong\Sigma_{\mathbf{v}_1}$. For this it is sufficient to show that $\{x\in\Sigma^\sharp_{\mathbf{v}_0}\mid\psi_{\mathbf{b}}(x)\in\mathbf{e}_0\}\cong\{x\in\Sigma_{\mathbf{v}_0}\mid\psi(\varphi(x))\in\mathbf{e}_0\}$ and that $\{z\in\Sigma^\sharp_{\mathbf{v}_1}\mid\psi^\vee_{\mathbf{b}}(z)\in\mathbf{e}_0\}\cong\{z\in\Sigma_{\mathbf{v}_1}\mid\psi^\vee(\varphi(z))\in\mathbf{e}_0\}$. Therefore we have to show that $\{x\in\Sigma^\sharp_{\mathbf{v}_0}\mid\psi_{\mathbf{b}}(x)\in\mathbf{e}_0\}\cong X^\Sigma_{\mathbf{e}_0}\bigcap\Sigma_{\mathbf{v}_0}$ and $\{z\in\Sigma^\sharp_{\mathbf{v}_1}\mid\psi^\vee_{\mathbf{b}}(z)\in\mathbf{e}_0\}\cong X^\Sigma_{\mathbf{e}_0}\bigcap\Sigma_{\mathbf{v}_1}$.

To prove statement for vertices $\mathbf{v}$ of type $\tau(\mathbf{v}) = 0$, we embed $X^{\Sigma}_{\mathbf{e}_0}$ into $Y^s_{\mathbf{b}} \subset \mathbb{P}$ as the set of points such that $\psi((x_1, x_2)) \in \mathbf{e}_0$ and $(\frac{x_0}{x_2})^{q+1} = -f_{\mathbf{e}_0}((x_1, x_2))$. We consider $X^{\Sigma}_{\mathbf{e}_0} \bigcap \Sigma_{\mathbf{v}_0}$ therefore as being the subset $\{x \in X^{\Sigma}_{\mathbf{e}_0} \mid d_{\mathbf{b}}(\psi((x_1, x_2)), \mathbf{v}_0) < 1\} = \{x \in X^{\Sigma}_{\mathbf{e}_0} \mid |\frac{\pi x_2}{x_1}| < 1\}$ of $Y^s_{\mathbf{b}}$.

Since $|\frac{\pi x_2}{x_1}| < 1$, one has that $f_{\mathbf{e}_0}((x_1, x_2)) = \frac{x_1}{x_2} \cdot \frac{1+(\frac{x_1}{x_2})^{(q-1)}}{1+(\frac{-\pi \cdot x_2}{x_1})^{(q-1)}} \equiv \frac{x_1}{x_2} + (\frac{x_1}{x_2})^q \mod \pi$. Therefore $x_0^{q+1} = -x_2^{q+1} f_{\mathbf{e}_0}((x_1, x_2)) \equiv -x_1^q x_2 - x_1 x_2^q \mod \pi$ holds. Since $x_1^q x_2 + x_1 x_2^q \equiv F_{\mathbf{b}}((x_1, x_2)) \mod \pi$, it follows that $X^{\Sigma}_{\mathbf{e}_0} \bigcap \Sigma_{\mathbf{v}_0} \cong \{x \in Y^s_{\mathbf{b}} \mid \psi_{\mathbf{b}}(x) \in \mathbf{e}_0, d_{\mathbf{b}}(\psi_{\mathbf{b}}(x), \mathbf{v}_0) < 1,\ x_0^{q+1} + F_{\mathbf{b}}((x_1, x_2)) = 0\} = \{x \in \Sigma^{\sharp}_{\mathbf{v}_0} \mid \psi_{\mathbf{b}}(x) \in \mathbf{e}_0\}$. This proves the proposition for the vertices $\mathbf{v}$ of type $\tau(\mathbf{v}) = 0$.

The statement of the proposition is proved similarly for the vertex $\mathbf{v}_1$ of type $\tau(\mathbf{v}_1) = 1$. One embeds $X^{\Sigma}_{\mathbf{e}_0}$ into $Y^{s\vee}_{\mathbf{b}} \subset \mathbb{P}^{\vee}$ as the set of points such that $\psi^{\vee}((z_1, z_2)) \in \mathbf{e}_0$ and $(\frac{z_0}{z_1})^{q+1} = -1/f_{\mathbf{e}_0}((z_1, z_2))$. Since $|\frac{z_1}{z_2}| < 1$ on $X^{\Sigma}_{\mathbf{e}_0} \bigcap \Sigma^{\sharp}_{\mathbf{v}_1}$, it follows that $-\pi \cdot z_0^{q+1} \equiv -(z_1^q \pi z_2 + z_1(\pi z_2)^q) \equiv -F^{\vee}_{\mathbf{b}}((z_1, z_2)) \mod \pi$ holds. From this the proposition follows. □

6.8 Proposition. *Let $g \in SU(2, L)$ and let $\mathbf{e} = g(\mathbf{e}_0) \in \mathbf{b}$ be an edge. Then the spaces $\Sigma^{\sharp}_{\mathbf{e}}$ and $\Sigma^{\sharp\vee}_{\mathbf{e}}$ are isomorphic.*

The isomorphism $\Sigma^{\sharp}_{\mathbf{e}} \cong \Sigma^{\sharp\vee}_{\mathbf{e}}$ is given by taking $z_1 = x_1$, $z_2 = -x_2$ and as z_0 the unique solution of $z_0^{q+1} = \frac{F_{\mathbf{b}}((x_1,x_2))F^{\vee}_{\mathbf{b}}((z_1,z_2))}{-\pi \cdot x_0^{q+1}}$ that satisfies $z_0 \equiv \frac{g^* x_1 g^* x_2}{x_0} \mod \pi$. The isomorphism does not depend on the choice of the element $g \in SU(2, L)$ such that $g(\mathbf{e}_0) = \mathbf{e}$.

Proof. The action of the group $SU(2, L)$ on $\mathbb{P}$ and $\mathbb{P}^{\vee}$ preserves the quadratic form $x_1 z_2 + x_2 z_1$. Therefore identifying $z_1 = x_1$ and $z_2 = -x_2$ on $\Sigma^{\sharp}_{\mathbf{e}}$ and $\Sigma^{\sharp\vee}_{\mathbf{e}}$ implies that $g^* z_1 = g^* x_1$ and $g^* z_2 = -g^* x_2$ holds. Then $1 < |\frac{g^* x_1}{g^* x_2}| = |\frac{g^* z_1}{g^* z_2}| < |\pi|$.

Once one has identified $\Sigma^{\sharp}_{\mathbf{e}}$ with $\Sigma^{\sharp\vee}_{\mathbf{e}}$, the equation $z_0^{q+1} = \frac{F_{\mathbf{b}}(x) F^{\vee}_{\mathbf{b}}(z)}{-\pi \cdot x_0^{q+1}}$ holds. We use this equation to derive a relation between the coordinates x_0 and z_0 that can be used to obtain the isomorphism between the two spaces. Since $1 < |\frac{g^* x_1}{g^* x_2}| < |\pi|$, we have $F_{\mathbf{b}}((x_1, x_2)) F^{\vee}_{\mathbf{b}}((z_1, z_2))/(-\pi) \equiv a_{\mathbf{v},\mathbf{b}}((x_1, x_2)) a_{\mathbf{v}'}((z_1, z_2))/(-\pi) \equiv$
$(g^* x_1 g^* x_2^q + g^* x_2 g^* x_1^q)(g^* x_1(-\pi g^* x_2)^q + (-\pi g^* x_2) g^* x_1^q)/(-\pi) \equiv (g^* x_1 g^* x_2)^{q+1} \mod \pi$.

Therefore we can identify the coordinate z_0 with the unique solution of the equation above such that $z_0 \equiv \frac{g^* x_1 g^* x_2}{x_0} \mod \pi$ as stated in the proposition.

This defines a bijection between the coordinates x_i, $i = 0, 1, 2$ on $\Sigma^{\sharp}_{\mathbf{e}}$ and the coordinates z_i, $i = 0, 1, 2$ on $\Sigma^{\sharp\vee}_{\mathbf{e}}$. Hence this gives an isomorphism between $\Sigma_{\mathbf{e}}^{\sharp}$ and $\Sigma_{\mathbf{e}}^{\sharp\vee}$.

If $g, g' \in SU(2,L)$ are elements such that $g(\mathbf{e}_0) = g'(\mathbf{e}_0) = \mathbf{e}$, then $|\frac{g'^*x_i}{g^*x_i}| \equiv 1 \mod \pi$ for $i = 1,2$. Therefore the choice of z_0 does not depend on the element $g \in SU(2,L)$ used in the proposition. Hence the identification $\Sigma_{\mathbf{e}}^{\sharp} \cong \Sigma_{\mathbf{e}}^{\sharp\vee}$ does not depend on the choice of the element $g \in SU(2,L)$. $\square$

6.9. Identifying coordinates in $\mathbb{P}$ and $\mathbb{P}^{\vee}$. The identifications of the coordinates x_i and z_i, $i = 1,2$ using the equation $x_1z_2 + x_2z_2 = 0$ can easily be done in advance for all of $\mathbb{P}$ and $\mathbb{P}^{\vee}$. Then $\mathbb{P}$ and $\mathbb{P}^{\vee}$ have the line $(0, x_1, x_2) \times (0, z_1, z_2)$ with the equation $x_1z_2 + x_2z_2 = 0$ in common. This line is preserved by the action of the group $U(2, L)$.

The resulting variety contains the blow up of $\mathbb{P}$ in the point $(x_0, 0, 0)$ and the blow up of $\mathbb{P}^{\vee}$ in $(z_0, 0, 0)$. These are the points that are fixed under the action of the group $U(1, L) \times U(2, L)$.

The exceptional line $x_1 = x_2 = 0$ of the blow up of $\mathbb{P}$ is identified with the ordinary line $(0, z_1, z_2) \subset \mathbb{P}^{\vee}$ and the exceptional line $z_1 = z_2 = 0$ of the blow up of $\mathbb{P}^{\vee}$ is identified with the ordinary line $(0, x_1, x_2) \subset \mathbb{P}$ by the equation $x_1z_2 + x_2z_2 = 0$. Since $(x_0, 0, 0) \notin Y_{\mathbf{b}}^{s}$ and $(z_0, 0, 0) \notin Y_{\mathbf{b}}^{s\vee}$, the spaces $Y_{\mathbf{b}}^{s} \subset \mathbb{P}$ and $Y_{\mathbf{b}}^{s\vee} \subset \mathbb{P}^{\vee}$ are not affected by the blow ups.

6.10 Proposition. *Let $\mathbf{e} \in \mathbf{b}$ be an edge and let $\mathbf{v}, \mathbf{v}' \in \mathbf{e}$ be the vertices of type $\tau(\mathbf{v}) = 0$ and $\tau(\mathbf{v}') = 1$. The analytical spaces $\Sigma_{\mathbf{v}}^{\sharp} \subset Y_{\mathbf{b}}^{s}$ and $\Sigma_{\mathbf{v}'}^{\sharp} \subset Y_{\mathbf{b}}^{s\vee}$ can be glued together by identifying the open admissable subspaces $\Sigma_{\mathbf{e}}^{\sharp} \subset \Sigma_{\mathbf{v}}^{\sharp}$ and $\Sigma_{\mathbf{e}}^{\sharp\vee} \subset \Sigma_{\mathbf{v}'}^{\sharp}$. Then the image of $\{x \in \Sigma_{\mathbf{v}}^{\sharp} \mid \psi_{\mathbf{b}}(x) \in \mathbf{e}\} \bigcup \{z \in \Sigma_{\mathbf{v}'}^{\sharp} \mid \psi_{\mathbf{b}}^{\vee}(z) \in \mathbf{e}\}$ in the resulting space is an affinoid isomorphic to $X_{\mathbf{e}}^{\Sigma}$.*

Proof. Let us define $\Sigma_{\mathbf{v}}^{\sharp}(\mathbf{e}) := \{x \in \Sigma_{\mathbf{v}}^{\sharp} \mid \psi_{\mathbf{b}}(x) \in \mathbf{e}\} = \{x \in Y_{\mathbf{b}}^{s} \mid \psi_{\mathbf{b}}(x) \in \mathbf{e}, \psi_{\mathbf{b}}(x) \neq \mathbf{v}', x_0^{q+1} = -F_{\mathbf{b}}((x_1, x_2))\} \subset \Sigma_{\mathbf{v}}^{\sharp} \subset Y_{\mathbf{b}}^{s}$. Similarly, we define $\Sigma_{\mathbf{v}'}^{\sharp}(\mathbf{e})$ as $\Sigma_{\mathbf{v}'}^{\sharp}(\mathbf{e}) := \{z \in \Sigma_{\mathbf{v}'}^{\sharp} \mid \psi_{\mathbf{b}}^{\vee}(z) \in \mathbf{e}\} = \{z \in Y_{\mathbf{b}}^{s\vee} \mid \psi_{\mathbf{b}}^{\vee}(z) \in \mathbf{e}, \psi_{\mathbf{b}}^{\vee}(z) \neq \mathbf{v}, \pi \cdot z_0^{q+1} = F_{\mathbf{b}}^{\vee}((z_1, z_2))\} \subset \Sigma_{\mathbf{v}'}^{\sharp} \subset Y_{\mathbf{b}}^{s\vee}$. Then $\Sigma_{\mathbf{e}}^{\sharp} \subset \Sigma_{\mathbf{v}}^{\sharp}(\mathbf{e})$ and $\Sigma_{\mathbf{e}}^{\sharp\vee} \subset \Sigma_{\mathbf{v}'}^{\sharp}(\mathbf{e})$. In fact $\Sigma_{\mathbf{e}}^{\sharp} = \{x \in \Sigma_{\mathbf{v}}^{\sharp}(\mathbf{e}) \mid 1 < |\frac{g^*x_1}{g^*x_2}| < |\pi|\}$ and $\Sigma_{\mathbf{e}}^{\sharp\vee} = \{z \in \Sigma_{\mathbf{v}'}^{\sharp}(\mathbf{e}) \mid 1 < |\frac{g^*z_1}{g^*z_2}| < |\pi|\}$.

Let us glue $\Sigma_{\mathbf{v}}^{\sharp}(\mathbf{e})$ and $\Sigma_{\mathbf{v}'}^{\sharp}(\mathbf{e})$ by using the isomorphism between $\Sigma_{\mathbf{e}}^{\sharp}$ and $\Sigma_{\mathbf{e}}^{\sharp\vee}$. In the proof above that $\Sigma_{\mathbf{v}} \cong \Sigma_{\mathbf{v}}^{\sharp}$, we have already shown that $\Sigma_{\mathbf{v}}^{\sharp}(\mathbf{e}) \cong \Sigma_{\mathbf{v}} \bigcap X_{\mathbf{e}}^{\Sigma}$ and that $\Sigma_{\mathbf{v}'}^{\sharp}(\mathbf{e}) \cong \Sigma_{\mathbf{v}'} \bigcap X_{\mathbf{e}}^{\Sigma}$. On $\Sigma_{\mathbf{e}}^{\sharp} \cong \Sigma_{\mathbf{e}}^{\sharp\vee}$ the equation $(\frac{x_0}{z_0})^{q+1}/(-\pi) = \frac{F_{\mathbf{b}}(x)}{F_{\mathbf{b}}^{\vee}(z)}$ holds. In $\Sigma_{\mathbf{e}}^{\sharp}$ seen as a subspace of $\Sigma_{\mathbf{v}'}^{\sharp}(\mathbf{e}) \subset \Sigma_{\mathbf{v}'}^{\sharp}$, one can multiplying the left side by $\pi \cdot z_0^{q+1}$ and the right side by $F_{\mathbf{b}}^{\vee}(z)$. Then one obtains the equation $x_0^{q+1} + F_{\mathbf{b}}(x) = 0$ that defines the space $\Sigma^\sharp_{\mathbf{v}}$. Similarly, in $\Sigma^\sharp_{\mathbf{e}}$ seen as a subspace of $\Sigma^\sharp_{\mathbf{v}}(\mathbf{e}) \subset \Sigma^\sharp_{\mathbf{v}}$, one can use the equation $-\pi \cdot (\frac{z_0}{x_0})^{q+1} = \frac{F^\vee_{\mathbf{b}}(z)}{F_{\mathbf{b}}(x)}$ and multiply the left side by x_0^{q+1} and the right side by $F_{\mathbf{b}}(x)$ to obtain the equation that defines the space $\Sigma^\sharp_{\mathbf{v}'}$. Therefore the space constructed is indeed isomorphic to the affinoid space $X^\Sigma_{\mathbf{e}}$.

The covering $\{\Sigma^\sharp_{\mathbf{v}}(\mathbf{e}), \Sigma^\sharp_{\mathbf{e}} \cong \Sigma^{\sharp\vee}_{\mathbf{e}}, \Sigma^\sharp_{\mathbf{v}'}(\mathbf{e})\}$ is an open admissable covering of the affinoid space isomorphic to $X^\Sigma_{\mathbf{e}} \subset \Sigma$. The covering $\{\Sigma^\sharp_{\mathbf{v}}, \Sigma^\sharp_{\mathbf{e}} \cong \Sigma^{\sharp\vee}_{\mathbf{e}}, \Sigma^\sharp_{\mathbf{v}'}\}$ is an open admissable covering of the space obtained by glueing $\Sigma^\sharp_{\mathbf{v}}$ and $\Sigma^\sharp_{\mathbf{v}'}$ along $\Sigma^\sharp_{\mathbf{e}} \cong \Sigma^{\sharp\vee}_{\mathbf{e}}$. □

6.11 Theorem. *Let $\Sigma^\sharp := \bigcup_{\mathbf{v}\in\mathbf{b}} \Sigma^\sharp_{\mathbf{v}}/\sim$. Here $\sim$ denotes the equivalence relation obtained by applying the isomorphisms $\Sigma^\sharp_{\mathbf{e}} \cong \Sigma^{\sharp\vee}_{\mathbf{e}}$ for all edges $\mathbf{e} \in \mathbf{b}$. Then $\Sigma^\sharp$ is a well-defined rigid analytic variety and $\Sigma^\sharp \cong \Sigma$.*

Proof. In prop. 6.10 above it is proved that for the vertices $\mathbf{v}, \mathbf{v}' \in \mathbf{e}$ the spaces $\Sigma^\sharp_{\mathbf{v}}$ and $\Sigma^\sharp_{\mathbf{v}'}$ can be glued by applying the isomorphism $\Sigma^\sharp_{\mathbf{e}} \cong \Sigma^{\sharp\vee}_{\mathbf{e}}$. Since the spaces $\Sigma^\sharp_{\mathbf{e}}$ are disjoint for the edges $\mathbf{e} \in \mathbf{b}$, one can use all the identifications $\Sigma^\sharp_{\mathbf{e}} \cong \Sigma^{\sharp\vee}_{\mathbf{e}}$ simultaneously to obtain a well-defined rigid analytic variety $\Sigma^\sharp = \bigcup_{\mathbf{v}\in\mathbf{b}} \Sigma^\sharp_{\mathbf{v}}/\sim$. Since this space consists of an affinoid isomorphic to $X^\Sigma_{\mathbf{e}}$ for each edge $\mathbf{e}$ in $\mathbf{b}$, the space $\Sigma^\sharp$ is isomorphic to Σ. This concludes the proof of the theorem. □

6.12 Corollary. *The following statements hold:*

i) $\Gamma_{\mathbf{b}}$ *acts linearly on* $\bigcup_{\mathbf{v}\in\mathbf{b},\tau(\mathbf{v})=0} \Sigma^\sharp_{\mathbf{v}} \subset Y^s_{\mathbf{b}}$ *through the coordinates* x_i, $i = 0, 1, 2$.

ii) $\Gamma_{\mathbf{b}}$ *acts linearly on* $\bigcup_{\mathbf{v}\in\mathbf{b},\tau(\mathbf{v})=1} \Sigma^\sharp_{\mathbf{v}} \subset Y^{s\vee}_{\mathbf{b}}$ *through the coordinates* z_i, $i = 0, 1, 2$.

iii) We have a map $\psi^{\Sigma^\sharp}_{\mathbf{b}} : \Sigma^\sharp \to \mathbf{b}$*, such that* $\psi^{\Sigma^\sharp}_{\mathbf{b}}(x) = \psi_{\mathbf{b}}(x)$ *for* $x \in \Sigma^\sharp_{\mathbf{v}}$ *with* $\tau(\mathbf{v}) = 0$ *and* $\psi^{\Sigma^\sharp}_{\mathbf{b}}(z) = \psi^\vee_{\mathbf{b}}(z)$ *for* $z \in \Sigma^\sharp_{\mathbf{v}}$ *with* $\tau(\mathbf{v}) = 1$.

Proof. The linear action of $\Gamma_{\mathbf{b}}$ follows directly from the construction of the spaces and the fact that the infinite sums $F_{\mathbf{b}}((x_1, x_2))$ and $F^\vee_{\mathbf{b}}((z_1, z_2))$ are $\Gamma_{\mathbf{b}}$-invariant.

Statement (iii) of the corollary follows directly from the fact that the maps $\psi_{\mathbf{b}}$ and $\psi^\vee_{\mathbf{b}}$ coincide on $\Sigma^\sharp_{\mathbf{e}}$ and $\Sigma^{\sharp\vee}_{\mathbf{e}}$ for $\mathbf{e} \in \mathbf{b}$. □

6.2 Another equivariant embedding

In this subsection we give a different embedding of the admissable subspaces $\Sigma_{\mathbf{v}} \subset \Sigma$ into $\mathbb{P} \cong \mathbb{P}^2_L$ for the vertices $\mathbf{v} \in \mathbf{b}$ of type $\tau(\mathbf{v}) = 0$. It does not explicitly use the fact that Σ is a covering of Ω_1. It uses explicitly the fact that the component of the reduction of Σ belonging to the vertex $\mathbf{v} \in \mathbf{b}$ is a hermitian curve. This embedding will later be used to construct a space $\mathcal{Y}$ on which discrete subgroups of $PU(3, L)$ act with proper quotients.

6.13. Polynomials for the vertices $\mathbf{v} \in \mathbf{b}$ of type $\tau(\mathbf{v}) = 0$. For each vertex $\mathbf{v} \in \mathbf{b}$ of type $\tau(\mathbf{v}) = 0$, we define a homogeneous polynomial $b_{\mathbf{v}}(x)$ of degree $q + 1$ for $x \in Y^s_{\mathbf{b}}$, such that $b_{\mathbf{v}}(x) \equiv x_0^{q+1} + a_{\mathbf{v},\mathbf{b}}((x_1, x_2)) \mod \pi$. The polynomials satisfy the following condition:

For all $\gamma \in \Gamma_{\mathbf{b}}$ and all $\mathbf{v} \in \mathbf{b}$ of type $\tau(\mathbf{v}) = 0$ one has $b_{\gamma(\mathbf{v})}(x) = \gamma^* b_{\mathbf{v}}(x)$.

From now on we change the notation a little. We view the polynomials $a_{\mathbf{v},\mathbf{b}}(z)$ and function $F^\vee_{\mathbf{b}}(z)$ as defined for points $z \in Y^{s\vee}_{\mathbf{b}}$ and the function $F_{\mathbf{b}}(x)$ as defined for points $x \in Y^s_{\mathbf{b}}$.

6.14 Lemma. *Let $\mathbf{v} \in \mathbf{b}$ be a vertex of type $\tau(\mathbf{v}) = 0$ and let $\mathbf{e} \ni \mathbf{v}$ be an edge. Let $A \in \mathbf{b}$ be an apartment that contains the edge $\mathbf{e}$. Let $x \in \Sigma^\sharp_{\mathbf{v}} \subset Y^s_{\mathbf{b}}$ be a point such that $\psi_{\mathbf{b}}(x) \in \mathbf{e}$. Let x_0, x_1, x_2 be the coordinates belonging to the apartment A. Then $v(\frac{x_1x_2}{x_0^2}) = \frac{q-1}{q+1} \cdot d_{\mathbf{b}}(\mathbf{v}, \psi_{\mathbf{b}}(x))$ and $1 \geq |\frac{x_1x_2}{x_0^2}| > |\pi|^{(q-1)/(q+1)}$. Moreover, $|\frac{x_1x_2}{x_0^2}| = 1$ if and only if $\psi_{\mathbf{b}}(x)$ is the vertex $\mathbf{v}$ of type $\tau(\mathbf{v}) = 0$*

Proof. Let $\mathbf{v} \in A$ be the vertex of type $\tau(\mathbf{v}) = 0$ closest to $\psi_{\mathbf{b}}(x)$. Then $F_{\mathbf{b}}((x_1, x_2)) \equiv a_{\mathbf{v},\mathbf{b}}((x_1, x_2)) \mod \pi$ and $|x_0^{q+1}| = |a_{\mathbf{v},\mathbf{b}}((x_1, x_2))|$. Hence $v(\frac{x_0^{q+1}}{(x_1x_2)^{(q+1)/2}}) = v(\frac{a_{\mathbf{v},\mathbf{b}}((x_1,x_2))}{(x_1x_2)^{(q+1)/2}}) = -\frac{q-1}{2} \cdot d_{\mathbf{b}}(\mathbf{v}, \psi_{\mathbf{b}}(x))$ by lemma 6.3(i) above. Therefore $v(\frac{x_0^2}{(x_1x_2)}) = -\frac{q-1}{q+1} \cdot d_{\mathbf{b}}(\mathbf{v}, \psi_{\mathbf{b}}(x))$. Since $0 \leq d_{\mathbf{b}}(\mathbf{v}, \psi_{\mathbf{b}}(x)) < 1$ the second statement follows. Moreover, if and only if $d_{\mathbf{b}}(\mathbf{v}, \psi_{\mathbf{b}}(x)) = 0$ does $|\frac{x_1x_2}{x_0^2}| = 1$ hold. $\square$

6.15. Analytical subspaces of $Y^s_{\mathbf{b}}$ and $Y^{s\vee}_{\mathbf{b}}$ for vertices and edges in $\mathbf{b}$. Let $\mathbf{v} \in \mathbf{b}$ be a vertex of type $\tau(\mathbf{v})$. If $\tau(\mathbf{v}) = 0$, then we take $\Sigma_{\mathbf{v},\mathbf{b}} := \{x \in Y^s_{\mathbf{b}} \mid d_{\mathbf{b}}(\psi_{\mathbf{b}}(x), \mathbf{v}) < 1,\ b_{\mathbf{v}}(x) = 0\}$. If $\tau(\mathbf{v}) = 1$, then $\Sigma_{\mathbf{v},\mathbf{b}} := \Sigma^\sharp_{\mathbf{v}} \subset Y^{s\vee}_{\mathbf{b}}$.

For edges $\mathbf{e} \in \mathbf{b}$ we define two analytical spaces $\Sigma_{\mathbf{e},\mathbf{b}} \subset Y^s_{\mathbf{b}}$ and $\Sigma^\vee_{\mathbf{e},\mathbf{b}} \subset Y^{s\vee}_{\mathbf{b}}$. Let $\Sigma_{\mathbf{e},\mathbf{b}} := \{x \in \Sigma_{\mathbf{v},\mathbf{b}} \mid \psi_{\mathbf{b}}(x) \in \mathbf{e}, \psi_{\mathbf{b}}(x) \neq \mathbf{v}\}$, where $\mathbf{v} \in \mathbf{e}$ is the vertex of type $\tau(\mathbf{v}) = 0$. Let $\Sigma^\vee_{\mathbf{e},\mathbf{b}} := \Sigma^{\sharp\vee}_{\mathbf{e}}$.

6.16 Proposition. *Let* $\mathbf{v} \in \mathbf{b}$ *be a vertex of type* $\tau(\mathbf{v}) = 0$*. Then* $\Sigma_{\mathbf{v},\mathbf{b}} \cong \Sigma^\sharp_{\mathbf{v}}$*.*

Proof. The space $\Sigma^\sharp_{\mathbf{v}}$ is defined as $\Sigma^\sharp_{\mathbf{v}} = \{x \in Y^s_{\mathbf{b}} \mid d_{\mathbf{b}}(\psi_{\mathbf{b}}(x), \mathbf{v}) < 1, x_0^{q+1} = -F_{\mathbf{b}}((x_1, x_2))\}$. Furthermore, $F_{\mathbf{b}}((x_1, x_2)) \equiv a_{\mathbf{v},\mathbf{b}}((x_1, x_2)) \mod \pi$. Therefore $x_0^{q+1} + a_{\mathbf{v},\mathbf{b}}((x_1, x_2)) \cdot (F_{\mathbf{b}}((x_1, x_2))/a_{\mathbf{v},\mathbf{b}}((x_1, x_2))) = 0$ holds with $F_{\mathbf{b}}((x_1, x_2))/a_{\mathbf{v},\mathbf{b}}((x_1, x_2)) \equiv 1 \mod \pi$. For $x \in \Sigma_{\mathbf{v},\mathbf{b}}$, one has that $b_{\mathbf{v}}(x) \equiv x_0^{q+1} + a_{\mathbf{v},\mathbf{b}}((x_1, x_2)) \equiv 0 \mod \pi$ holds. From this the isomorphism follows. $\square$

6.17 Proposition. *Let* $g \in SU(2, L)$ *and let* $\mathbf{e} = g(\mathbf{e}_0) \in \mathbf{b}$ *be an edge. Then the spaces* $\Sigma_{\mathbf{e},\mathbf{b}}$ *and* $\Sigma^\vee_{\mathbf{e},\mathbf{b}}$ *are isomorphic.*

The isomorphism $\Sigma_{\mathbf{e},\mathbf{b}} \cong \Sigma^\vee_{\mathbf{e},\mathbf{b}}$ *is given by taking* $z_1 = x_1$*,* $z_2 = -x_2$ *and as* z_0 *the solution of* $z_0^{q+1} = \frac{a_{\mathbf{v},\mathbf{b}}((x_1,x_2))F^\vee_{\mathbf{b}}((z_1,z_2))}{-\pi \cdot x_0^{q+1}}$ *that satisfies* $z_0 \equiv \frac{g^* x_1 g^* x_2}{x_0} \mod \pi$*. The identifications given do not depend on the choice of the element* $g \in SU(2, L)$ *such that* $g(\mathbf{e}_0) = \mathbf{e}$*.*

Proof. Similar to the proof that $\Sigma^\sharp_{\mathbf{e}}$ and $\Sigma^{\sharp\vee}_{\mathbf{e}}$ are isomorphic, once one observes that $a_{\mathbf{v},\mathbf{b}}((x_1, x_2)) \equiv F_{\mathbf{b}}((x_1, x_2)) \mod \pi$ on $\Sigma_{\mathbf{v},\mathbf{b}} \supset \Sigma_{\mathbf{e},\mathbf{b}}$. $\square$

6.18 Theorem. *Let* $\Sigma_{\mathbf{b}} := \bigcup_{\mathbf{v}\in\mathbf{b}} \Sigma_{\mathbf{v},\mathbf{b}}/\sim$*. Here* $\sim$ *denotes the equivalence relation obtained by applying the isomorphisms* $\Sigma_{\mathbf{e},\mathbf{b}} \cong \Sigma^\vee_{\mathbf{e},\mathbf{b}}$ *for all edges* $\mathbf{e} \in \mathbf{b}$*. Then* $\Sigma_{\mathbf{b}}$ *is a well-defined rigid analytic variety and* $\Sigma_{\mathbf{b}} \cong \Sigma$*.*

Proof. Similar to the proof that $\Sigma^\sharp$ is a well-defined rigid analytic variety isomorphic to Σ. $\square$

6.19. Other possible simplifications. One can again simplify the construction somewhat by identifying the coordinates x_1, x_2 of $\mathbb{P}$ with the coordinates z_1, z_2 of $\mathbb{P}^\vee$ through the relation $x_1 z_2 + x_2 z_1 = 0$.

A more significant simplification can be obtained by also defining polynomials $b_{\mathbf{v}}(z) \equiv -\pi \cdot z_0^{q+1} + a_{\mathbf{v},\mathbf{b}}(z) \mod \pi$ for the vertices $\mathbf{v} \in \mathbf{b}$ of type $\tau(\mathbf{v}) = 1$. These polynomials can then be used to define for vertices $\mathbf{v} \in \mathbf{b}$ of type $\tau(\mathbf{v}) = 1$ a analytical space $\Sigma'_{\mathbf{v},\mathbf{b}} := \{z \in Y^{s\vee}_{\mathbf{b}} \mid b_{\mathbf{v}}(z) = 0, d_{\mathbf{b}}(\psi^\vee_{\mathbf{b}}(z), \mathbf{v}) < 1\}$. The proofs in this case are again quite similar to the ones presented above. For our purposes in the sections below the construction of $\Sigma_{\mathbf{b}}$ as presented here is sufficient.

7 Stable points in the projective plane

In this section we describe in some detail the points in the projective plane that are stable for all maximal K-split tori of the group $U(3, L)$. In particular, we describe a $U(3, L)$-equivariant map from the set of stable points to the building $\mathcal{B}$. The results presented here have been proved in [L-V] and [V].

The map will be used extensively to construct the uniformising space and its compactification in the sections that follow. In particular, the map will allow us to define suitable affinoid spaces and pure affinoid coverings that correspond to simplices in the building $\mathcal{B}$.

7.1. Stable and semistable points. Let $\mathbb{P}$ and $\mathbb{P}^\vee$ be distinct projective planes $\mathbb{P}^2_L$ with coordinates x_0, x_1, x_2 and z_0, z_1, z_2, respectively. Let the group $U(3, L)$ act linearly on $\mathbb{P}$ and $\mathbb{P}^\vee$ preserving the hermitian form h and $h^\vee$, respectively.

For a maximal K-split torus in $U(3, L)$ we use the linearisation that is the restriction to the torus of the $U(3, L)$-linearisation of $\mathcal{O}(1)$. The homogeneous torus invariants are generated by the monomials x_0^2 and $x_1 x_2$ in $\mathbb{P}$ and by the monomials z_0^2 and $z_1 z_2$ in $\mathbb{P}^\vee$

Let $Y^{ss} \subset \mathbb{P}$ and $Y^{ss\vee} \subset \mathbb{P}^\vee$ denote the open analytical subspaces that contain the points that are semistable for all maximal K-split tori in $U(3, L)$. Similarly, we denote by $Y^s \subset \mathbb{P}$ and $Y^{s\vee} \subset \mathbb{P}^\vee$ the open analytical subspaces consisting of the points that are stable for all maximal K-split tori in $U(3, L)$. Then $Y^{ss} := \mathbb{P}^2_L - \{y \in \mathbb{P}^2(L) \mid h(y, y) = 0\}$ and $Y^s := \{x \in \mathbb{P}^2_L \mid \forall (y \in \mathbb{P}^2(L)$ such that $h(y, y) = 0)\ h(x, y) \neq 0\}$. Similar descriptions hold for $Y^{ss\vee}$ and $Y^{s\vee}$.

7.2. Criterion for semistability. We define a function $r(x)$ on $\mathbb{P}^2_L$ involving torus invariants that can be used to define the space Y^{ss} .

Let $A \subset \mathcal{B}$ be an apartment with coordinates x_0, x_1, x_2 and let $g \in U(3, L)$. Then we define:

$$r_{gA,A}(x) := \begin{cases} 0 & \text{if } x_0^2 = x_1 x_2 = 0 \\ max\{|g^* x_0^2|, |g^* x_1 g^* x_2|\}/max\{|x_0^2|, |x_1 x_2|\} & \text{if } max\{|x_0^2|, |x_1 x_2|\} \neq 0. \end{cases}$$

Then $r_{gA,A}(x)$ is well-defined for $x \in \mathbb{P}^2_L$.

Let $r(x) := \inf\{r_{gA,A}(x) \mid g \in U(3, L)\}$ for $x \in \mathbb{P}^2_L$. Then $r(x) > 0$ if and only if $x \in Y^{ss}$ and there exists an apartment $gA \in \mathcal{B}$ such that $r_{gA,A}(x) = r(x)$ (See [P-V] §3.6).

7.3. The interval of semistability. Let $\mathbf{v} \in \mathcal{B}$ be a vertex of type $\tau(\mathbf{v}) = 0$. To the vertex $\mathbf{v}$ belongs an equivalence class of L°-modules $[M_{\mathbf{v}}]$. Let $M_{\mathbf{v}}$ be the module $< e_0, e_1, e_2 >$.

Let $\mathbb{C}_p^\circ$ be the ring of integers of the completion of the algebraic closure of K. Let $\mathcal{M}_{\mathbf{v}}$ be the $\mathbb{C}_p^\circ$-module $\mathcal{M}_{\mathbf{v}} := M_{\mathbf{v}} \otimes \mathbb{C}_p^\circ$. For a rational point $u \in \mathcal{B}$, there exists an apartment $A \ni \mathbf{v}, u$ and a torus element $s \in S$, where S is the torus belonging to the apartment A, such that $u = s \cdot \mathbf{v}$. To the point $u = s \cdot \mathbf{v} \in \mathcal{B}$ we associate the $\mathbb{C}_p^\circ$-module $\mathcal{M}_u := s \cdot \mathcal{M}_{\mathbf{v}}$.

The parahoric group $P_u \subset U(3, L)$ acts on $\mathcal{M}_u$. The equivalence class $[\mathcal{M}_u]$ of $\mathbb{C}_p^\circ$-modules does not depend on the choice of the apartment A, the torus element s or the vertex $\mathbf{v}$.

If for all apartments $A \ni u$ the reduction $\overline{x} \in \mathbb{P}(\mathcal{M}_u \otimes \ell)$ of x is semistable for the reduction $S \otimes \ell$ of the torus S that belongs to the apartment A, then we say that x *is semistable in the reduction for $U(3, L)$ at $u \in \mathcal{B}$*.

Let $x \in Y^{ss}$ be a point. The *interval of semistability $I(x)$ of x for the group $U(3, L)$* is the closure in $\mathcal{B}$ of the set of points $u \in \mathcal{B}(\mathbb{Q})$ such that x is semistable in the reduction for the group $U(3, L)$ at u:

$I(x) := \overline{\{u \in \mathcal{B}(\mathbb{Q}) \mid x \text{ is semistable in the reduction for } U(3, L) \text{ at } u\}} \subset \mathcal{B}$.

The interval of semistability $I_{\mathbf{b}}(x)$ of $x \in Y_{\mathbf{b}}^{ss}$ for the group $U(2, L)$ acting on the building $\mathbf{b}$ is defined analogously. The point $x \in Y_{\mathbf{b}}^{ss}$ *is semistable in the reduction for the group $U(2, L)$ at $u \in \mathbf{b}$*, if for all apartments $A \subset \mathbf{b}$, $A \ni u$, the reduction $\overline{x} \in \mathbb{P}(\mathcal{M}_u \otimes \ell)$ of x is semistable for the reduction $S \otimes \ell$ of the torus S that belongs to the apartment A. Then:

$I_{\mathbf{b}}(x) := \overline{\{u \in \mathbf{b}(\mathbb{Q}) \mid x \text{ is semistable in the reduction for } U(2, L) \text{ at } u\}} \subseteq \mathbf{b}$.

The subsets $I_{\mathbf{b}}(x) \subseteq \mathbf{b}$ and $I(x) \subset \mathcal{B}$ are convex for a point $x \in Y_{\mathbf{b}}^{ss}$ and $x \in Y^{ss}$, respectively. The interval $I(x)$ is bounded if and only if $x \in Y^s$. (See [V] cor. 4.10.) If $x \in Y^{ss} - Y^s$, then the interval of semistability $I(x)$ is not bounded. As an example take the point $(x_0, 0, 0) \in \mathbb{P}_L^2$ that is stabilised by the group $PU(2, L)$ belonging to $\mathbf{b} \subset \mathcal{B}$. Then $I(x) = \mathbf{b}$.

For the action of $U(2, L)$ on $Y_{\mathbf{b}}^{ss\vee}$ and $U(3, L)$ on $Y^{ss\vee}$ one analogously defines intervals of semistability $I_{\mathbf{b}}^\vee(z)$ and $I^\vee(z)$ for points z in $Y_{\mathbf{b}}^{ss\vee}$ and $Y^{ss\vee}$, respectively.

For later use we recall some results from [V]:

7.4 Proposition. *Let $x \in Y^{ss}$ be a point. Then there exists a $PU(2, L)$-building $\mathbf{b} \subset \mathcal{B}$ such that $I(x) = I_{\mathbf{b}}(x)$. In particular, $I(x) \subseteq \mathbf{b}$.*

Proof. See [V] theorem 6.2. □

7.5 Proposition. *Let $x \in Y^{ss}$ and let $\mathbf{b} \subset \mathcal{B}$ be a $PU(2,L)$-building. Then the following two statements hold:*

i) If the intersection $I(x) \bigcap \mathbf{b}$ is non-empty, then $I_{\mathbf{b}}(x) = I(x) \bigcap \mathbf{b}$.

ii) If the intersection $I(x) \bigcap \mathbf{b}$ is empty, then $I_{\mathbf{b}}(x) = \{\mathbf{v}\}$. Here $\mathbf{v} \in \mathbf{b}$ is the unique vertex such that $d_{\mathcal{B}}(\mathbf{v}, I(x)) = d_{\mathcal{B}}(\mathbf{b}, I(x))$.

Proof. This is a direct consequence of [V] prop. 6.5, where a similar statement is proved for an apartment $A \subset \mathcal{B}$, instead of a $PU(2,L)$-building $\mathbf{b} \subset \mathcal{B}$. $\square$

7.6 Lemma. *Let $x \in Y^s_{\mathbf{b}}$ and let $A \subset \mathbf{b}$ be an apartment such that $\psi_{\mathbf{b}}(x) \in A$. Let $\rho_{\mathbf{b}}(x) := min\{v(\frac{x_0^2}{x_1 x_2}), 0\}$, where the x_i, $i = 0,1,2$ are the coordinates of $\mathbb{P}^2_L$ corresponding to A. Then $I_{\mathbf{b}}(x) := \{u \in \mathbf{b} \mid d_{\mathbf{b}}(\psi_{\mathbf{b}}(x), u) \leq -\rho_{\mathbf{b}}(x)\}$.*

Proof. For the convenience of the reader we give a proof here, even though this has been proved in [V] prop. 5.6.

Let $A \subset \mathbf{b}$ be an apartment containing $\psi_{\mathbf{b}}(x)$. Let x_0, x_1, x_2 be the coordinates of $\mathbb{P}^2_L$, such that the torus S belonging to A acts diagonally. Since $|\frac{g^* x_i}{x_i}| = 1$, $i = 1,2$ and $g^* x_0 = x_0$ for all elements $g \in P_{\psi_{\mathbf{b}}(x)} \bigcap SU(2,L)$, the value of $\rho_{\mathbf{b}}(x)$ does not depend on the apartment $A \ni \psi_{\mathbf{b}}(x)$ used.

Let $t \in S$ be an element such that $|t^* x_1| = |t^* x_2|$ holds for the point $x \in Y^s_{\mathbf{b}}$. We replace the coordinates x_i by $t^* x_i$ for $i = 0,1,2$. Now the coordinates x_i are coordinates of $\mathbb{P}^2_{\mathbb{C}_p}$ instead of $\mathbb{P}^2_L$.

Let $s \in S$ be the diagonal element $diag(1, s_1, s_2)$ w.r.t. the coordinates x_i, $i = 0,1,2$ of $\mathbb{P}^2_{\mathbb{C}_p}$. The reduction $\bar{x}$ is semistable for the torus $S \otimes \ell$ at $s \cdot \psi_{\mathbf{b}}(x)$ if and only if $|s^* x_1^2| \leq max\{|x_0^2|, |x_1 x_2|\}$ and $|s^* x_2^2| \leq max\{|x_0^2|, |x_1 x_2|\}$. Furthermore, $|s^* x_1^2|, |s^* x_2^2| \leq max\{|x_0^2|, |x_1 x_2|\}$ if and only if $\rho_{\mathbf{b}}(x) \leq v(\frac{s_1}{s_2}) \leq -\rho_{\mathbf{b}}(x)$.

Since $d_{\mathbf{b}}(s \cdot \psi_{\mathbf{b}}(x), \psi_{\mathbf{b}}(x)) = v(\frac{s_1}{s_2})$, the reduction $\bar{x}$ of x is semistable for the torus $S \otimes \ell$ at $s \cdot \psi_{\mathbf{b}}(x) \in A \in \mathbf{b}$ if and only if $d_{\mathbf{b}}(s \cdot \psi_{\mathbf{b}}(x), \psi_{\mathbf{b}}(x)) \leq -\rho_{\mathbf{b}}(x)$.

Since $g \in P_{s \cdot \psi_{\mathbf{b}}(x)} \bigcap U(2,L)$ acts only on the coordinates x_1 and x_2, the point x is also semistable in the reduction at $s \cdot \psi_{\mathbf{b}}(x)$ for the torus $gSg^{-1} \otimes \ell$. Hence x is semistable in the reduction at $s \cdot \psi_{\mathbf{b}}(x)$ for all apartments $A \ni s \cdot \psi_{\mathbf{b}}(x)$.

Therefore x is semistable in the reduction for the group $U(2,L)$ at the point $u \in \mathbf{b}$ if and only if $d_{\mathbf{b}}(u, \psi_{\mathbf{b}}(x)) \leq -\rho_{\mathbf{b}}(x)$. $\square$

7.7. A $U(3,L)$-equivariant map $\psi_{\mathcal{B}} : Y^s \longrightarrow \mathcal{B}$. Let $x \in Y^s$ be a point. Since $Y^s \subset Y^s_{\mathbf{b}}$, the image $\psi_{\mathbf{b}}(x) \in \mathbf{b}$ is well-defined for any $PU(2,L)$-building $\mathbf{b} \subset \mathcal{B}$.

Let $A \subset \mathcal{B}$ be an apartment such that $r_{A,A}(x) = r(x)$. Then A is contained in a unique $PU(2,L)$-building $\mathbf{b} \subset \mathcal{B}$. Therefore $I_{\mathbf{b}}(x)$ and $\psi_{\mathbf{b}}(x)$ are well-defined. Let us consider the set $b_{\psi_{\mathbf{b}}(x)}$ containing the $PU(2,L)$-buildings $\mathbf{b}' \ni \psi_{\mathbf{b}}(x)$. Then $\{\rho_{\mathbf{b}'}(x) \mid \mathbf{b}' \in b_{\psi_{\mathbf{b}}(x)}\}$ obtains its infinum for some $PU(2,L)$-building $\mathbf{b}'' \in b_{\psi_{\mathbf{b}}(x)}$. Indeed, the compact group $P_{\psi_{\mathbf{b}}(x)} \subset U(3,L)$ that stabilises the point $\psi_{\mathbf{b}}(x) \in \mathcal{B}$ acts transitively on the set $b_{\psi_{\mathbf{b}}(x)}$ and the function $\rho_{\mathbf{b}'}(x), \mathbf{b}' \in b_{\psi_{\mathbf{b}}(x)}$ is continuous. Without loss of generality we may assume that $\mathbf{b} = \mathbf{b}''$. Then $I(x) = I_{\mathbf{b}}(x) \subset \mathbf{b}$ and we define $\psi_{\mathcal{B}}(x) := \psi_{\mathbf{b}}(x)$. Therefore $\psi_{\mathcal{B}}(x)$ is the center of the interval of the interval of semistability of x (See [V] def. 6.4). A map $\psi^{\vee}_{\mathcal{B}} : Y^{s\vee} \to \mathcal{B}$ is defined analogously.

8 An admissable open subspace of $\Sigma_{\mathbf{b}}$

An open admissable subspace $\Sigma^{\circ}_{\mathbf{b}} \subset \Sigma_{\mathbf{b}}$ is defined. We define and describe in some detail an open admissable subspace $\Sigma^{\circ}_{\mathbf{v},\mathbf{b}} \subset \Sigma_{\mathbf{v},\mathbf{b}}$. The space $\Sigma^{\circ}_{\mathbf{b}}$ is obtained by glueing the spaces $\Sigma^{\circ}_{\mathbf{v},\mathbf{b}}$ for the vertices $\mathbf{v} \in \mathbf{b}$. The space $\Sigma^{\circ}_{\mathbf{b}}$ will be used to construct the uniformising space in the next section.

8.1. Open admissable subspaces of $\Sigma_{\mathbf{b}}$ belonging to vertices v and edges e in b. For a $PU(2,L)$-building $\mathbf{b} \subset \mathcal{B}$, we consider the space $\Sigma_{\mathbf{b}}$ that belongs to the group $PU(2,L)$ that acts on $\mathbf{b} \subset \mathcal{B}$. For each vertex $\mathbf{v} \in \mathbf{b}$ we define an admissable open subspace $\Sigma^{\circ}_{\mathbf{v},\mathbf{b}} \subset \Sigma_{\mathbf{v},\mathbf{b}}$.

If the vertex $\mathbf{v}$ is of type $\tau(\mathbf{v}) = 0$, then $\Sigma^{\circ}_{\mathbf{v},\mathbf{b}}$ is defined as $\Sigma^{\circ}_{\mathbf{v},\mathbf{b}} := \Sigma_{\mathbf{v},\mathbf{b}} \bigcap \{x \in Y^s \mid d_{\mathcal{B}}(\psi_{\mathcal{B}}(x), \mathbf{v}) < 1\}$. If $\tau(\mathbf{v}) = 1$, then $\Sigma^{\circ}_{\mathbf{v},\mathbf{b}} := \Sigma_{\mathbf{v},\mathbf{b}} \bigcap \{z \in Y^{s\vee} \mid d_{\mathcal{B}}(\psi^{\vee}_{\mathcal{B}}(z), \mathbf{v}) < 1\}$.

Similarly, we define for edges $\mathbf{e} \ni \mathbf{b}$ two spaces. Let $\mathbf{v}_i \in \mathbf{e}$ be the vertex of type $\tau(\mathbf{v}_i) = i$ for $i = 0,1$. Then $\Sigma^{\circ}_{\mathbf{e},\mathbf{b}} := \Sigma_{\mathbf{e},\mathbf{b}} \bigcap \{x \in Y^s \mid \psi_{\mathcal{B}}(x) \in \mathbf{e}, \psi_{\mathcal{B}}(x) \neq \mathbf{v}_0\}$ and $\Sigma^{\circ\vee}_{\mathbf{e},\mathbf{b}} := \Sigma^{\vee}_{\mathbf{e},\mathbf{b}} \bigcap \{z \in Y^{s\vee} \mid \psi^{\vee}_{\mathcal{B}}(z) \in \mathbf{e}, \psi^{\vee}_{\mathcal{B}}(z) \neq \mathbf{v}_1\}$.

8.2 Lemma. *Let $\mathbf{v} \in \mathbf{b}$ be a vertex of type $\tau(\mathbf{v}) = 0$ and let $x \in \Sigma^{\circ}_{\mathbf{v},\mathbf{b}}$. Then the following statements hold:*

i) If $\psi_{\mathcal{B}}(x) \neq \mathbf{v}$, then $\mathbf{v} \notin I(x)$.

ii) If $\psi_{\mathcal{B}}(x) = \mathbf{v}$, then $I(x) = \{\mathbf{v}\}$.

iii) Let $\mathbf{b}' \subset \mathcal{B}$ *be a* $PU(2, L)$*-building that contains the vertex* $\mathbf{v}$*.*

a) If $\psi_{\mathcal{B}}(x) \notin \mathbf{b}'$*, then* $I_{\mathbf{b}'}(x) = \{\mathbf{v}\}$*.*

b) If $\psi_{\mathcal{B}}(x) \in \mathbf{b}'$*, then* $I_{\mathbf{b}'}(x) = I(x)$*.*

Proof. To prove the first two statements of the lemma, we consider an element $g \in P_{\mathbf{v}}$ such that $\psi_{\mathcal{B}}(x) \in g(A)$ and $I(x) \subset g(\mathbf{b})$. Since $d_{\mathcal{B}}(\psi_{\mathcal{B}}(x), \mathbf{v}) < 1$, such an element g exists.

Now $b_{\mathbf{v}}(x) = g^*x_0^{q+1} + g^*x_1 \cdot g^*x_2^q + g^*x_2 \cdot g^*x_1^q + \pi \cdot f(x) = 0$, where $f(x)$ is a homogeneous polynomial of degree $q+1$. Without loss of generality we may assume that $|\frac{g^*x_1}{g^*x_2}| \leq 1$. Then $(\frac{g^*x_0}{g^*x_2})^{q+1} + \frac{g^*x_1}{g^*x_2} + (\frac{g^*x_1}{g^*x_2})^q + \pi \cdot \frac{f(x)}{g^*x_2^{q+1}} = 0$. Since $d_{\mathcal{B}}(\psi_{\mathcal{B}}(x), \mathbf{v}) < 1$ and $\psi_{g(\mathbf{b})}(x) = \psi_{\mathcal{B}}(x)$, we have $1 \geq |\frac{g^*x_1}{g^*x_2}| > |\pi|$. Therefore $|\frac{g^*x_0}{g^*x_2}|^{q+1} = |\frac{g^*x_1}{g^*x_2}|$ holds, unless $|\frac{g^*x_1}{g^*x_2}| = 1$ and $|\frac{g^*x_1}{g^*x_2} + (\frac{g^*x_1}{g^*x_2})^q| < 1$ holds.

In the latter case $\psi_{\mathcal{B}}(x) = \mathbf{v}$ and $\frac{g^*x_1}{g^*x_2} \equiv \omega \mod \pi$ for some $\omega \in L^\circ$ such that $\omega^q = -\omega$. We will show that this cannot occur. There exists an element $h \in SU(3, L)$ such that $h^*g^*x_1 = g^*x_1 - \omega g^*x_2$, $h^*g^*x_2 = g^*x_2$, $h^*g^*x_0 = g^*x_0$. Then $|\frac{h^*g^*x_1 h^*g^*x_2}{g^*x_1 g^*x_2}| < 1$. It follows that $\psi_{\mathcal{B}}(x) \notin gA$. In particular, $\psi_{\mathcal{B}}(x) \neq \mathbf{v}$. This contradicts our assumptions. Hence this cannot occur.

We conclude that $|\frac{g^*x_0}{g^*x_2}|^{q+1} = |\frac{g^*x_1}{g^*x_2}|$ holds. Multiplying both sides with $|\frac{g^*x_2}{g^*x_1}|^{(q+1)/2}$ gives $|\frac{g^*x_0^2}{g^*x_1 g^*x_2}|^{(q+1)/2} = |\frac{g^*x_2}{g^*x_1}|^{(q-1)/2}$. Otherwise stated $\rho_{g(\mathbf{b})}(x) = \frac{q-1}{q+1} \cdot v(\frac{g^*x_2}{g^*x_1}) = -\frac{q-1}{q+1} \cdot d_{\mathcal{B}}(\psi_{\mathcal{B}}(x), \mathbf{v})$.

Therefore if $\psi_{\mathcal{B}}(x) \neq \mathbf{v}$, then $\mathbf{v} \notin I_{g(\mathbf{b})}(x)$ and $I(x) = I_{g(\mathbf{b})}(x)$. If $\psi_{\mathcal{B}}(x) = \mathbf{v}$, then $I(x) = I_{g(\mathbf{b})}(x) = \{\mathbf{v}\}$. This proves statements (i) and (ii) of the lemma.

Statement (iii)a follows from statement (i) and the fact that $\mathbf{v} \in \mathbf{b}'$ is the vertex closest to $I(x)$ if $\psi_{\mathcal{B}}(x) \notin \mathbf{b}'$. Statement (iii)b is a direct consequence of statements (i) and (ii) of the lemma. □

8.3. Isotropic points. Let $\mathbf{v} \in \mathbf{b} \subset \mathcal{B}$ be a vertex of type $\tau(\mathbf{v}) = 0$ and let $[M_{\mathbf{v}}]$ be the corresponding equivalence class of L°-modules. Let $a \in \mathbb{P}(M_{\mathbf{v}})$ be an isotropic point such that the reduction of a is not contained in the reduction of the $\mathbb{P}^1_L$ given by $x_0 = 0$ belonging to $\mathbf{b}$. Then $a = (a_0, a_1, a_2)$ with $|\frac{a_0^2}{a_1 a_2}| = 1$.

The closed ball of radius r around a in $\Sigma_{\mathbf{v},\mathbf{b}}$ is defined by $B(a, r) := \{x \in \Sigma_{\mathbf{v},\mathbf{b}} \mid |\frac{h(x,a)}{a_0 \cdot x_0}| \leq r\}$. Since $x_0 \neq 0$ for $x \in \Sigma_{\mathbf{v},\mathbf{b}}$, this is well-defined.

For a vertex $\mathbf{v} \in \mathbf{b} \subset \mathcal{B}$ we define the set $Iso(\mathbf{v},\mathbf{b})$ as $Iso(\mathbf{v},\mathbf{b}) := \{a \in \mathbb{P}^2(L) \mid h(a,a) = 0, |\frac{a_0^2}{a_1 a_2}| = 1\}$.

8.4 Proposition. *Let $\mathbf{b} \subset \mathcal{B}$ be a $PU(2,L)$-building and let $\mathbf{v} \in \mathbf{b}$ be a vertex and let $\mathbf{e} \in \mathbf{b}$ be an edge. Then the following statements hold:*

i) If $\tau(\mathbf{v}) = 0$, then $\Sigma^\circ_{\mathbf{v},\mathbf{b}} = \Sigma_{\mathbf{v},\mathbf{b}} - \bigcup_{a \in Iso(\mathbf{v},\mathbf{b})} B(a, |\pi|)$.

ii) If $\tau(\mathbf{v}) = 1$, then $\Sigma^\circ_{\mathbf{v},\mathbf{b}} = \Sigma_{\mathbf{v},\mathbf{b}}$.

iii) $\Sigma^\circ_{\mathbf{e},\mathbf{b}} = \Sigma_{\mathbf{e},\mathbf{b}}$ and $\Sigma^{\circ\vee}_{\mathbf{e},\mathbf{b}} = \Sigma^\vee_{\mathbf{e},\mathbf{b}}$.

Proof. Let us prove statement (i) of the proposition. We first show that the points $x \in \Sigma_{\mathbf{v},\mathbf{b}}$ such that $\psi_{\mathbf{b}}(x) \neq \mathbf{v}$ are contained in $\Sigma^\circ_{\mathbf{v},\mathbf{b}}$. If $x \in \Sigma_{\mathbf{v},\mathbf{b}}$, then $\rho_{\mathbf{b}}(x) := min\{v(\frac{x_0^2}{x_1 x_2}), 0\} = v(\frac{x_0^2}{x_1 x_2}) = -\frac{q-1}{q+1} \cdot d_{\mathbf{b}}(\mathbf{v}, \psi_{\mathbf{b}}(x))$.

Since by assumption $\psi_{\mathbf{b}}(x) \neq \mathbf{v}$, the value of $\rho_{\mathbf{b}}(x)$ is non-zero. Therefore $I_{\mathbf{b}}(x) = I(x) \bigcap \mathbf{b}$ holds. The convexity of $I(x)$ and the fact that $\mathbf{v} \notin I_{\mathbf{b}}(x)$ imply that $I(x) = I_{\mathbf{b}}(x)$ holds. Therefore $\psi_{\mathcal{B}}(x) = \psi_{\mathbf{b}}(x)$ and $d_{\mathcal{B}}(\mathbf{v}, \psi_{\mathcal{B}}(x)) < 1$. Hence $x \in \Sigma^\circ_{\mathbf{v},\mathbf{b}}$. One verifies that $x \notin B(a, |\pi|)$ for all $a \in Iso(\mathbf{v},\mathbf{b})$.

Let us now consider the case where $\psi_{\mathbf{b}}(x) = \mathbf{v}$. Let $a \in Iso(\mathbf{v},\mathbf{b})$ be the isotropic point $a = (a_0, a_1, a_2)$. Take $\alpha = \overline{(\frac{a_0}{a_2})}$ and $\beta = \overline{(\frac{a_2}{a_2})}$. Then one can define an element $g_a \in P_{\mathbf{v}} \subset SU(3,L)$ as follows: $g_a^* x_1 = x_1 + \alpha \cdot x_0 + \beta \cdot x_2$, $g_a^* x_2 = x_2$, $g_a^* x_0 = x_0 - \overline{\alpha} \cdot x_2$. Then $(g_a^* x_0, g_a^* x_1, g_a^* x_2) = (x_0 - \frac{a_0 x_2}{a_2}, \frac{h(x,a)}{\overline{a_2}}, x_2)$. In particular, for the point a, one has: $(g_a^* x_0, g_a^* x_1, g_a^* x_2) = (0, 0, a_2)$. The set of elements $\{g_a \in P_{\mathbf{v}} \mid a \in Iso(\mathbf{v},\mathbf{b})\}$ acts transitively on the halfapartments that start in the vertex $\mathbf{v}$ and that are not contained in $\mathbf{b}$.

Let us assume that $\psi_{\mathbf{b}}(x) = \mathbf{v}$ and that $x \notin Y^s$. Then $x \notin \Sigma^\circ_{\mathbf{v},\mathbf{b}}$ and we have to show that $x \in B(a, |\pi|)$ for some $a \in Iso(\mathbf{v},\mathbf{b})$.

Since $x \notin Y^s$, there exist a point $y \in \mathbb{P}^2(L)$ such that $h(y,y) = 0$ and $h(x,y) = 0$. If $y \in Iso(\mathbf{v},\mathbf{b})$, then $g_y^* x_1 = \frac{h(x,y)}{\overline{y_2}} = 0$. Hence $x \in B(y, |\pi|)$. If $y \notin Iso(\mathbf{v},\mathbf{b})$, then $|y_0^2| < |y_1 y_2|$ and $|y_1 x_2 + y_2 x_1| < 1$. In particular, $\psi_{\mathbf{b}}(x) \neq \mathbf{v}$. This contradicts our assumptions. Hence this cannot occur and $x \in Y^s$ must hold.

Let us now assume that $\psi_{\mathbf{b}}(x) = \mathbf{v}$, $x \in Y^s$ and moreover, that $d_{\mathcal{B}}(\psi_{\mathcal{B}}(x), \mathbf{v}) \geq 1$. Then $x \notin \Sigma^\circ_{\mathbf{v},\mathbf{b}}$ and we have to show that $x \in B(a, |\pi|)$ for some $a \in Iso(\mathbf{v},\mathbf{b})$. There exists an element $g_a \in SU(3,L)$ such that the apartment $g_a A$ contains $\psi_{\mathcal{B}}(x)$. Then $I_{g_a(\mathbf{b})}(x) = I(x) \bigcap g_a(\mathbf{b}) \subseteq I(x)$. We can choose the element g_a in such a way that the apartment $g_a A$ contains a point $r \in I(x)$, that has distance $R = max\{d_{\mathcal{B}}(u, \psi_{\mathcal{B}}(x)) \mid u \in I(x)\}$, the radius of $I(x)$ to the center $\psi_{\mathcal{B}}(x)$ of $I(x)$ and distance $d_{\mathcal{B}}(\mathbf{v}, \psi_{\mathcal{B}}(x)) + R$ to the vertex $\mathbf{v}$. Then $d_{\mathcal{B}}(\psi_{g_a(\mathbf{b})}(x), \mathbf{v}) \geq d_{\mathcal{B}}(\psi_{\mathcal{B}}(x), \mathbf{v})$ holds. In particular, $d_{\mathcal{B}}(\psi_{g_a(\mathbf{b})}(x), \mathbf{v}) \geq 1$. Therefore $v(\frac{g_a^* x_1}{g_a^* x_2}) = v(\frac{h(x,a)}{\overline{a_2} x_2}) \geq 1$. Hence $x \in B(a, |\pi|)$ and $x \notin \Sigma^\circ_{\mathbf{v},\mathbf{b}}$.

Next we show that if $\psi_{\mathbf{b}}(x) = \mathbf{v}$ and $d_{\mathcal{B}}(\psi_{\mathcal{B}}(x), \mathbf{v})) < 1$, then for all $a \in \mathit{Iso}(\mathbf{v}, \mathbf{b})$ the point $x \notin B(a, |\pi|)$. Indeed, since $d_{\mathcal{B}}(\psi_{\mathcal{B}}(x), \mathbf{v})) < 1$, for an element g_a we have either $I(x) \bigcap g_a(\mathbf{b}) = \emptyset$ and $I_{g_a(\mathbf{b})}(x) = \mathbf{v}$ or $I_{g_a(\mathbf{b})}(x) = I(x)$ and $\psi_{g_a(\mathbf{b})}(x) = \psi_{\mathcal{B}}(x)$. In both cases $0 \leq v(\frac{g_a^* x_1}{g_a^* x_2}) = v(\frac{h(x,a)}{\overline{a_2} x_2}) < 1$ holds. In particular, $x \notin B(a, |\pi|)$. This concludes the proof of statement (i) of the proposition.

Let us now prove statement (ii). Let $\tau(\mathbf{v}) = 1$ and let $z \in \Sigma_{\mathbf{v},\mathbf{b}}$. Then $v(\frac{\pi z_0^{q+1}}{(\pi z_1 z_2)^{(q+1)/2}}) = v(\frac{a_{\mathbf{v},\mathbf{b}}((z_1,z_2))}{(\pi z_1 z_2)^{(q+1)/2}}) = -\frac{q-1}{2} \cdot d_{\mathbf{b}}(\mathbf{v}, \psi^\vee_{\mathbf{b}}(z))$. Therefore $v(\frac{z_0^2}{\pi z_1 z_2}) = -\frac{2}{q+1} - \frac{q-1}{q+1} \cdot d_{\mathbf{b}}(\mathbf{v}, \psi^\vee_{\mathbf{b}}(z))$. So $v(\frac{z_0^2}{z_1 z_2}) = \frac{q-1}{q+1} - \frac{q-1}{q+1} \cdot d_{\mathbf{b}}(\mathbf{v}, \psi^\vee_{\mathbf{b}}(z)) > 0$ holds, since $d_{\mathbf{b}}(\mathbf{v}, \psi^\vee_{\mathbf{b}}(z)) < 1$. From this one concludes that $I^\vee_{\mathbf{b}}(z) = \{\psi^\vee_{\mathbf{b}}(z)\}$ holds. Therefore $\psi^\vee_{\mathcal{B}}(z) = \psi^\vee_{\mathbf{b}}(z)$ holds and $z \in \Sigma^\circ_{\mathbf{v},\mathbf{b}}$. Hence $\Sigma^\circ_{\mathbf{v},\mathbf{b}} = \Sigma_{\mathbf{v},\mathbf{b}}$ if $\tau(\mathbf{v}) = 1$.

Let us now prove statement (iii). If $x \in \Sigma_{\mathbf{e},\mathbf{b}}$ then $\psi_{\mathbf{b}}(x) \in \mathbf{e}$ and $\psi_{\mathbf{b}}(x) \neq \mathbf{v}$. Then $I(x) = I_{\mathbf{b}}(x)$ and $\psi_{\mathcal{B}}(x) = \psi_{\mathbf{b}}(x) \in \mathbf{e}$. Hence $x \in \Sigma^\circ_{\mathbf{e},\mathbf{b}}$. Therefore $\Sigma^\circ_{\mathbf{e},\mathbf{b}} = \Sigma_{\mathbf{e},\mathbf{b}}$ holds. For $z \in \Sigma^\vee_{\mathbf{e},\mathbf{b}}$ one again verifies that $\psi^\vee_{\mathcal{B}}(z) = \psi^\vee_{\mathbf{b}}(z)$ holds. Therefore $\Sigma^{\circ\vee}_{\mathbf{e},\mathbf{b}} = \Sigma^\vee_{\mathbf{e},\mathbf{b}}$ holds.

This concludes the proof of the proposition. □

8.5 Proposition. *Let* $\Sigma^\circ_{\mathbf{b}} := \bigcup_{\mathbf{v} \in \mathbf{b}} \Sigma^\circ_{\mathbf{v},\mathbf{b}} / \sim$*, where* $\sim$ *denotes the equivalence relation obtained by applying the identifications* $\Sigma^\circ_{\mathbf{e},\mathbf{b}} \cong \Sigma^{\circ,\vee}_{\mathbf{e},\mathbf{b}}$ *for all edges* $\mathbf{e} \in \mathbf{b}$*. Then the following four statements hold:*

i) The space $\Sigma^\circ_{\mathbf{b}}$ *is a well-defined rigid analytical variety.*

ii) The space $\Sigma^\circ_{\mathbf{b}}$ *is an open admissable subspace of* $\Sigma_{\mathbf{b}}$*.*

iii) $\Sigma^\circ_{\mathbf{b}} = \Sigma_{\mathbf{b}} - \bigcup_{\mathbf{v} \in \mathbf{b}, \tau(\mathbf{v})=0} \bigcup_{c \in \mathit{Iso}(\mathbf{v},\mathbf{b})} B(c, |\pi|)$.

Proof. Statement (i) is clear from the construction. The second statement follows from the fact that the spaces $\Sigma^\circ_{\mathbf{v},\mathbf{b}} \subset \Sigma_{\mathbf{v},\mathbf{b}}$ are open admissable subspaces. The third statement follows from the description of the spaces $\Sigma^\circ_{\mathbf{v},\mathbf{b}}$ given in the proposition above. □

8.6 Proposition. *Let* $\mathbf{v} \in \mathbf{b}$ *be a vertex of type* $\tau(\mathbf{v})$*. Then:*

i) If $\tau(\mathbf{v}) = 0$*, then* $\Sigma^\circ_{\mathbf{v},\mathbf{b}} = \Sigma_{\mathbf{v},\mathbf{b}} \bigcap \{x \in Y^s \mid d_{\mathcal{B}}(\psi_{\mathcal{B}}(x), \mathbf{b}) < 1\}$.

ii) If $\tau(\mathbf{v}) = 1$, then $\Sigma^\circ_{\mathbf{v},\mathbf{b}} = \Sigma_{\mathbf{v},\mathbf{b}} \bigcap \{z \in Y^{s\vee} \mid d_{\mathcal{B}}(\psi^\vee_{\mathcal{B}}(z), \mathbf{b}) < 1\}$.

Proof. Let us prove the first statement of the proposition. For a point $x \in \Sigma^\circ_{\mathbf{v},\mathbf{b}}$, the inequality $d_{\mathcal{B}}(\psi_{\mathcal{B}}(x), \mathbf{b}) \leq d_{\mathcal{B}}(\psi_{\mathcal{B}}(x), \mathbf{v}) < 1$ holds. Hence the inclusion $\Sigma^\circ_{\mathbf{v},\mathbf{b}} \subset \{x \in Y^s \mid d_{\mathcal{B}}(\psi_{\mathcal{B}}(x), \mathbf{b}) < 1\}$ holds.

Now let $x \in \Sigma_{\mathbf{v},\mathbf{b}} - \Sigma^\circ_{\mathbf{v},\mathbf{b}}$. Then $x \in B(a, |\pi|)$ for some isotropic point $a \in Iso(\mathbf{v}, \mathbf{b})$. The isotropic point a corresponds to an edge $\mathbf{e} \ni \mathbf{v}$ that is not contained in $\mathbf{b}$. In particular, if $x \in Y^s$, then the path from $\psi_{\mathcal{B}}(x)$ to the vertex $\mathbf{v}$ contains the edge $\mathbf{e}$. Therefore the vertex $\mathbf{v} \in \mathbf{b}$ is such that the equality $d_{\mathcal{B}}(\psi_{\mathcal{B}}(x), \mathbf{b}) = d_{\mathcal{B}}(\psi_{\mathcal{B}}(x), \mathbf{v})$ holds. It follows that $\Sigma^\circ_{\mathbf{v},\mathbf{b}} = \Sigma_{\mathbf{v},\mathbf{b}} \bigcap \{x \in Y^s \mid d_{\mathcal{B}}(\psi_{\mathcal{B}}(x), \mathbf{b}) < 1\}$ holds.

The second statement is straightforward, since $\Sigma^\circ_{\mathbf{v},\mathbf{b}} = \Sigma_{\mathbf{v},\mathbf{b}}$ if $\tau(\mathbf{v}) = 1$. This concludes the proof of the proposition. □

9 The uniformising space

Let $\Gamma \subset PU(3, L)$ be a discrete co-compact subgroup that preserves an almost complete transversal covering $\mathcal{T}$ of $PU(2, L)$-buildings. In this section the spaces $\Sigma^\circ_{\mathbf{b}}$ for $\mathbf{b} \in \mathcal{T}$ are glued together into a space $\mathcal{Y}^\circ$.

In §8 a space $\Sigma^\circ_{\mathbf{b}}$ was constructed by glueing the spaces $\Sigma^\circ_{\mathbf{v},\mathbf{b}}$ for the vertices $\mathbf{v} \in \mathbf{b}$. We now choose the equations that define the spaces $\Sigma^\circ_{\mathbf{v},\mathbf{b}}$ with $\mathbf{v} \in \mathbf{b}$ and $\mathbf{b} \in \mathcal{T}$ in a Γ-invariant way. Then $\Sigma^\circ_{\mathbf{v},\mathbf{b}} = \Sigma^\circ_{\mathbf{v},\mathbf{b}'}$ holds, if $\mathbf{b}, \mathbf{b}' \in \mathcal{T}$ are such that $\mathbf{v} \in \mathbf{b}, \mathbf{b}'$. Since the edges $\mathbf{e} \in \mathcal{B}$ are contained in at most one building $\mathbf{b} \in \mathcal{T}$, we can now simultaneously apply the identifications $\Sigma^\circ_{\mathbf{e},\mathbf{b}} \cong \Sigma^{\circ\vee}_{\mathbf{e},\mathbf{b}}$ for all edges $\mathbf{e} \in \mathbf{b}$ and $\mathbf{b} \in \mathcal{T}$. The result is a space $\mathcal{Y}^\circ = \bigcup_{\mathbf{b} \in \mathcal{T}} \Sigma^\circ_{\mathbf{b}} / \sim_{\mathcal{T}}$. Here $\sim_{\mathcal{T}}$ is the equivalence relation that identifies $\Sigma^\circ_{\mathbf{v},\mathbf{b}}$ with $\Sigma^\circ_{\mathbf{v},\mathbf{b}'}$ for vertices $\mathbf{v} \in \mathbf{b}, \mathbf{b}'$ with $\mathbf{b}, \mathbf{b}' \in \mathcal{T}$.

The embedding of $\sum_{\mathbf{v} \in \mathbf{b}, \tau(\mathbf{v})=0} \Sigma^\circ_{\mathbf{v},\mathbf{b}}$ into $\mathbb{P}$ and of $\sum_{\mathbf{v} \in \mathbf{b}, \tau(\mathbf{v})=1} \Sigma^\circ_{\mathbf{v},\mathbf{b}}$ into $\mathbb{P}^\vee$ is such that $\Gamma \cap H_{\mathbf{b}}$ acts linearly on both. These linear actions can now be extended to $\sum_{\mathbf{b} \in \mathcal{T}} \sum_{\mathbf{v} \in \mathbf{b}, \tau(\mathbf{v})=0} \Sigma^\circ_{\mathbf{v},\mathbf{b}}$ and $\sum_{\mathbf{b} \in \mathcal{T}} \sum_{\mathbf{v} \in \mathbf{b}, \tau(\mathbf{v})=1} \Sigma^\circ_{\mathbf{v},\mathbf{b}}$. Therefore we have a well-defined action of Γ on the analytical space $\mathcal{Y}^\circ$. The action is discontinuous.

If the transversal covering $\mathcal{T}$ is complete, then the quotient $\mathcal{Y}^\circ / \Delta$ is a projective algebraic curve for normal subgroups $\Delta \subset \Gamma$ of finite index and without elements of finite order.

In general the quotients $\mathcal{Y}^\circ / \Delta$ are not proper. We will compactify such non-proper quotients in the next section.

9.1. Polynomials for the vertices contained in $|\mathcal{T}|$. Let $\Gamma \subset U(3, L)$ be a discrete co-compact subgroup that admits a Γ-invariant almost complete transversal covering $\mathcal{T}$ of $PU(2, L)$-buildings. We give extra conditions on the homogeneous polynomials $a_{\mathbf{v},\mathbf{b}}(z)$ and $b_{\mathbf{v}}(x)$ associated to the vertices $\mathbf{v} \in |\mathcal{T}| := \bigcup_{\mathbf{b}\in\mathcal{T}} \mathbf{b}$. These extra conditions allow us to glue the open admissable subsets $\Sigma^{\circ}_{\mathbf{b}} \subset \Sigma_{\mathbf{b}}$ for $\mathbf{b} \in \mathcal{T}$ together.

Let $\mathbf{b}_0 \in \mathcal{T}$ be an $PU(2, L)$-building. Let $\mathbf{v}_0 \in \mathbf{b}_0$ be the vertex corresponding to the equivalence class $[M_{\mathbf{v}_0}]$, $M_{\mathbf{v}_0} =< e_0, e_1, e_2 >$ and let $\mathbf{v}_1 \in \mathbf{b}_0$ be the vertex corresponding to the equivalence class $[M_{\mathbf{v}_1}]$, $M_{\mathbf{v}_1} =< e_0, e_1, \pi^{-1}e_2 >$. The homogeneous polynomial $b_{\mathbf{v}_0}(x)$ is such that $b_{\mathbf{v}_0}(x) \equiv x_0^{q+1} + x_1x_2^q + x_2x_1^q \mod \pi$. The homogeneous polynomial $a_{\mathbf{v}_1,\mathbf{b}_0}(z)$ is of degree $q+1$ in the coordinates z_1 and z_2 such that $a_{\mathbf{v}_1,\mathbf{b}_0}(z) \equiv z_1(\pi z_2)^q + z_1^q(\pi z_2) \mod \pi$ holds. Let $\mathbf{v} = g(\mathbf{v}_0)$. Then $b_{\mathbf{v}}(x) \equiv g^*b_{\mathbf{v}_0}(x) \mod \pi$. Let $\mathbf{v}' \in \mathbf{b} \in \mathcal{T}$ and let $g \in U(3, L)$ be such that $\mathbf{v}' = g(\mathbf{v}_1)$ and $\mathbf{b} = g(\mathbf{b}_0)$. Then the polynomial $a_{\mathbf{v}',\mathbf{b}}(z)$ is homogeneous of degree $q + 1$ in the coordinates g^*z_1 and g^*z_2 such that $a_{\mathbf{v}',\mathbf{b}}(z) \equiv g^*a_{\mathbf{v}_1,\mathbf{b}_0}(z) \mod \pi$ holds.

In this way we associate to each vertex $\mathbf{v} \in |\mathcal{T}| \subset \mathcal{B}$ of type $\tau(\mathbf{v}) = 0$ ($\tau(\mathbf{v}) = 1$) that is contained in a $PU(2, L)$-building $\mathbf{b} \in \mathcal{T}$ a homogeneous polynomial $b_{\mathbf{v}}(x)$ ($a_{\mathbf{v},\mathbf{b}}(z)$) of degree $q+1$ such that the following two conditions hold:

i) If $\tau(\mathbf{v}) = 0$, then $b_{\gamma(\mathbf{v})}(x) = \gamma^*b_{\mathbf{v}}(x)$ for all $\gamma \in \Gamma$.

ii) If $\tau(\mathbf{v}) = 1$, then $a_{\gamma(\mathbf{v}),\gamma(\mathbf{b})}(z) = \gamma^*a_{\mathbf{v},\mathbf{b}}(z)$ for all $\gamma \in \Gamma$.

Since $Char(K) = 0$, one can always replace a homogeneous polynomial $b_{\mathbf{v}}(x)$ by the homogeneous polynomial $\frac{1}{|\Gamma_{\mathbf{v}}|}\sum_{\gamma\in\Gamma_{\mathbf{v}}} \gamma^*b_{\mathbf{v}}(x)$ to obtain a polynomial that satisfies the condition (i).

9.2. Construction of the rigid analytic space $\mathcal{Y}^{\circ}$ for $|\mathcal{T}| := \bigcup_{\mathbf{b}\in\mathcal{T}} \mathbf{b}$. The space $\mathcal{Y}^{\circ}$ is defined by associating to each vertex $\mathbf{v} \in |\mathcal{T}|$ a analytic space $\mathcal{Y}_{\mathbf{v}}$ and glueing them together along open admissable subspaces $\mathcal{Y}_{\mathbf{e}}$ and $\mathcal{Y}^{\vee}_{\mathbf{e}}$ corresponding to edges $\mathbf{e} \in |\mathcal{T}|$.

The spaces $\mathcal{Y}_{\mathbf{v}}$ for $\mathbf{v} \in |\mathcal{T}|$ are defined as $\mathcal{Y}_{\mathbf{v}} := \Sigma^{\circ}_{\mathbf{v},\mathbf{b}}$. Here $\mathbf{b} \in \mathcal{T}$ is a $PU(2, L)$-building that contains the vertex $\mathbf{v}$. If $\tau(\mathbf{v}) = 1$, then the building $\mathbf{b} \ni \mathbf{v}$ is unique. If $\tau(\mathbf{v}) = 0$, then condition (i) on the polynomial $b_{\mathbf{v}}(x)$ ensures us that the space $\Sigma^{\circ}_{\mathbf{v},\mathbf{b}}$ does not depend on the choice of $\mathbf{b} \ni \mathbf{v}$. Clearly, $\mathcal{Y}_{\mathbf{v}} \subset Y^s$ if $\tau(\mathbf{v}) = 0$ and $\mathcal{Y}_{\mathbf{v}} \subset Y^{s\vee}$ if $\tau(\mathbf{v}) = 1$.

For the edges $\mathbf{e} \in |\mathcal{T}|$ we define the spaces $\mathcal{Y}_{\mathbf{e}} := \Sigma^{\circ}_{\mathbf{e},\mathbf{b}}$ and $\mathcal{Y}^{\vee}_{\mathbf{e}} := \Sigma^{\circ\vee}_{\mathbf{e},\mathbf{b}}$. Let $\mathbf{v}, \mathbf{v}' \in \mathbf{e}$ be the vertices of type $\tau(\mathbf{v}) = 0$ and type $\tau(\mathbf{v}') = 1$. Then $\mathcal{Y}_{\mathbf{e}} \subset \mathcal{Y}_{\mathbf{v}}$ and $\mathcal{Y}^{\vee}_{\mathbf{e}} \subset \mathcal{Y}_{\mathbf{v}'}$ are open admissable subspaces.

Since each edge $\mathbf{e} \in |\mathcal{T}|$ is contained in a unique $PU(2, L)$-building $\mathbf{b} \in \mathcal{T}$ and $\mathcal{Y}_{\mathbf{e}} \cong \Sigma_{\mathbf{e},\mathbf{b}}$ and $\mathcal{Y}^{\vee}_{\mathbf{e}} \cong \Sigma^{\vee}_{\mathbf{e},\mathbf{b}}$, we can identify $\mathcal{Y}_{\mathbf{e}} \cong \mathcal{Y}^{\vee}_{\mathbf{e}}$ by using the same identifications used to identify $\Sigma_{\mathbf{e},\mathbf{b}} \cong \Sigma^{\vee}_{\mathbf{e},\mathbf{b}}$.

9.3 Theorem. *Let $\mathcal{Y}^{\circ} := \bigcup_{\mathbf{v}\in|\mathcal{T}|} \mathcal{Y}_{\mathbf{v}}/\sim$, where $\sim$ is the equivalence relation that identifies $\mathcal{Y}_{\mathbf{e}} \cong \mathcal{Y}^{\vee}_{\mathbf{e}}$ for all edges $\mathbf{e} \in |\mathcal{T}|$. Then:*

i) The space $\mathcal{Y}^{\circ}$ is a well-defined rigid analytical space.

ii) $\mathcal{Y}^{\circ} = \bigcup_{\mathbf{b}\in\mathcal{T}} \Sigma^{\circ}_{\mathbf{b}}/\sim_{\mathcal{T}}$. Here $\sim_{\mathcal{T}}$ is the equivalence relation that identifies for all vertices $\mathbf{v} \in \mathcal{B}$ of type $\tau(\mathbf{v}) = 0$ the spaces $\Sigma^{\circ}_{\mathbf{v},\mathbf{b}} \cong \Sigma^{\circ}_{\mathbf{v},\mathbf{b}'}$ for all $\mathbf{b}, \mathbf{b}' \in \mathcal{T}$ with $\mathbf{v} \in \mathbf{b}, \mathbf{b}'$.

Proof. The identifications $\mathcal{Y}_{\mathbf{e}} = \Sigma_{\mathbf{e},\mathbf{b}} \cong \Sigma^{\vee}_{\mathbf{e},\mathbf{b}} = \mathcal{Y}^{\vee}_{\mathbf{e}}$ are well-defined for $\mathbf{e} \in \mathbf{b}$. Since each edge $\mathbf{e} \in \bigcup_{\mathbf{b}\in\mathcal{T}} \mathbf{b}$ is contained in a single building $\mathbf{b} \in \mathcal{T}$ and the spaces $\mathcal{Y}_{\mathbf{e}}$ are all disjoint, as are all the spaces $\mathcal{Y}^{\vee}_{\mathbf{e}}$, we can apply the identifications simultaneously for all edges $\mathbf{e} \in |\mathcal{T}|$. The result is a well-defined rigid analytic space $\mathcal{Y}^{\circ}$.

The second statement of the proposition follows from the fact that the identifications used to define the space $\mathcal{Y}^{\circ}$ when restricted to the spaces $\mathcal{Y}_{\mathbf{v}}$ with $\mathbf{v} \in \mathbf{b}$ for a $PU(2, L)$-building $\mathbf{b} \in \mathcal{T}$ define the space $\Sigma^{\circ}_{\mathbf{b}}$. $\square$

9.4 Proposition. *Let $\mathbf{v} \in |\mathcal{T}|$ be a vertex of type $\tau(\mathbf{v}) = 0$. Let $\mathbf{b}_i \in \mathcal{T}$, $i = 1, \ldots, s$ be the $PU(2, L)$-buildings that contain the vertex $\mathbf{v}$. Let $H_i \cong P(U(1, L) \times U(2, L)) \subset PU(3, L)$ be the stabiliser of $\mathbf{b}_i$ and let $a_i \in \mathbb{P}(L)$ be the point that is fixed by the group H_i for $i = 1, \ldots, s$. Then there exist open affine subvarieties $X_i \subset \mathbb{P}$, $i = 1, \ldots, s$ such that the following statements hold:*

i) $a_i \in X_i$ for $i = 1, \ldots, s$.

ii) The intersection of the reduction $X_i \otimes \ell$ and the hermitian curve given by $b_{\mathbf{v}}(x) \equiv 0 \mod \pi$ in $\mathbb{P}(M_{\mathbf{v}}) \otimes \ell$ is an open affine subvariety of the hermitian curve.

iii) Let $R_{\mathbf{v}} :\ \mathbb{P}(M_{\mathbf{v}}) \to \mathbb{P}(M_{\mathbf{v}} \otimes \ell)$ be the reduction map. Let $\mathbf{e} \in \mathbf{b}_i$ be an edge containing the vertex $\mathbf{v}$. We consider the space $\mathcal{Y}_{\mathbf{e}}$ as a analytic subspace of $\mathbb{P}(M_{\mathbf{v}}) = \mathbb{P}$ and the affine varieties $X_j \otimes \ell$ as subspaces of $\mathbb{P}(M_{\mathbf{v}} \otimes \ell)$ for $j = 1, \ldots, s$. Then the following statements hold:

a) $\mathcal{Y}_{\mathbf{e}} \bigcap R_{\mathbf{v}}^{-1}(X_i \otimes \ell) = \emptyset$.

b) $\mathcal{Y}_{\mathbf{e}} \subset R_{\mathbf{v}}^{-1}(X_j \otimes \ell)$ *for* $j \neq i$, $j = 1, \ldots, s$.

iv) Let $X_{\mathbf{v},\mathbf{b}_i} \subset \mathbb{P}(M_{\mathbf{v}}) = \mathbb{P}$ *be the open admissable analytical subspace* $X_{\mathbf{v},\mathbf{b}_i} := R_{\mathbf{v}}^{-1}(\bigcap_{j\neq i} X_j \otimes \ell)$ *if* $s > 1$. *If* $s = 1$, *then we define* $X_{\mathbf{v},\mathbf{b}_1} := \mathbb{P}(M_{\mathbf{v}})$. *Let* $\mathbf{e} \in |\mathcal{T}|$ *be an edge such that* $\mathbf{v} \in \mathbf{e}$. *Then the analytical space* $\mathcal{Y}_{\mathbf{e}}$ *is contained in the open admissable subspace* $X_{\mathbf{v},\mathbf{b}_i} \subset \mathbb{P}$ *if and only if* $\mathbf{e} \in \mathbf{b}_i$.

Proof. For the point $a_i = (x_0, 0, 0)$, we take as X_i the affine variety defined by $x_0 \neq 0$. Let $g_{i,j} \in P_{\mathbf{v}} \subset SU(3, L)$ be an element such that $g_{i,j}(a_i) = a_j$. Then $g_{i,j}(\mathbf{b}_i) = \mathbf{b}_j$ holds for $j \neq i$. We define X_j by $X_j := g_{i,j}(X_i)$ for $j \neq i$. We will show that the thus obtained varieties X_j satisfy the proposition.

By construction statement (i) of the proposition holds. On the hermitian curve given by $b_{\mathbf{v}}(x) \equiv 0 \mod \pi$ in $\mathbb{P}(M_{\mathbf{v}}) \otimes \ell$, the intersection is again given by $\overline{x_0} \neq 0$ and is affine. Therefore statement (ii) holds.

Let us now prove statement (iii) of the proposition. The reduction $R_{\mathbf{v}}(\mathcal{Y}_{\mathbf{e}}) \subset \mathbb{P}(M_{\mathbf{v}} \otimes \ell)$ consists of a single ℓ-valued isotropic point. This is the isotropic point that corresponds to the edge $\mathbf{e} \in \mathcal{B}$. We may assume that $R_{\mathbf{v}}(\mathcal{Y}_{\mathbf{e}}) = \{(0, x_1, 0)\}$ holds. Since $\overline{x_0} = 0$ holds on $R_{\mathbf{v}}(\mathcal{Y}_{\mathbf{e}})$, it follows that the intersection $R_{\mathbf{v}}(\mathcal{Y}_{\mathbf{e}}) \bigcap X_i \otimes \ell$ is empty. Hence statement iii)a of the proposition holds.

To prove statement iii)b, we take again the element $g_{i,j} \in P_{\mathbf{v}} \subset SU(3, L)$ such that $g_{i,j}(\mathbf{b}_i) = \mathbf{b}_j$ for $j \neq i$. Then $\overline{g_{i,j}^{*} x_0} \neq 0$ holds on $R_{\mathbf{v}}(\mathcal{Y}_{\mathbf{e}})$. Therefore $R_{\mathbf{v}}(\mathcal{Y}_{\mathbf{e}}) \subset X_j \otimes \ell$. From this statement iii)b of the proposition follows.

Statement (iv) of the proposition is a direct consequence of statement (iii). $\square$

9.5. Identification of coordinates in $\mathbb{P}$ and $\mathbb{P}^{\vee}$. The proposition above shows that around a vertex $\mathbf{v} \in |\mathcal{T}|$ of type $\tau(\mathbf{v}) = 0$, one can simplify the construction of the analytical space. On each of the open analytic subvarieties $X_{\mathbf{v},\mathbf{b}_i} \subset \mathbb{P}$ one can identify the coordinates of the line fixed by the group H_i in $\mathbb{P}$ with the coordinates of the line in $\mathbb{P}^{\vee}$ fixed by the group H_i. The identification of coordinates is given by a suitable translate of the equation $x_1 z_2 + x_2 z_1 = 0$. As a consequence one blows up the points in $\mathbb{P}$ and $\mathbb{P}^{\vee}$ that are fixed by the stabilisers H_i of the $PU(2, L)$-buildings $\mathbf{b}_i \in \mathcal{T}$ that contain the vertex $\mathbf{v}$ and identifies the exceptional lines in $\mathbb{P}$ (resp. $\mathbb{P}^{\vee}$) with the corresponding ordinary lines in $\mathbb{P}^{\vee}$ (resp. $\mathbb{P}$).

Let us now briefly discuss the situation for two $PU(2,L)$-buildings $\mathbf{b}_1, \mathbf{b}_2 \ni \mathbf{v}$ that do not intersect transversally at $\mathbf{v}$. Then they have an edge $\mathbf{e} \ni \mathbf{v}$ in common. Let the affine subvarieties $X_1 \subset \mathbb{P}$ and $X_2 \subset \mathbb{P}$ be defined as above. The isotropic point $R_{\mathbf{v}}(\mathcal{Y}_{\mathbf{e}})$ that corresponds to the edge $\mathbf{e}$ is neither contained in $X_1 \otimes \ell$ nor in $X_2 \otimes \ell$. Therefore $\mathcal{Y}_{\mathbf{e}}$ is not contained in $R_{\mathbf{v}}^{-1}(X_1 \otimes \ell)$ nor in $R_{\mathbf{v}}^{-1}(X_2 \otimes \ell)$. One needs to use another affine subvariety $Z_{\mathbf{e}} \subset \mathbb{P}(M_{\mathbf{v}}) = \mathbb{P}$ such that $\mathcal{Y}_{\mathbf{e}} \subset R_{\mathbf{v}}^{-1}(Z_{\mathbf{e}} \otimes \ell)$. On the open analytic space $R_{\mathbf{v}}^{-1}(Z_{\mathbf{e}} \otimes \ell)$ that contains the space $\mathcal{Y}_{\mathbf{e}}$ one must make a choice of which coordinates to use, i.e. those belonging to the building $\mathbf{b}_1$ or to the building $\mathbf{b}_2$. Therefore one has two choices for the coordinates that can be used to define the analytic spaces $\mathcal{Y}_{\mathbf{e}}^{\vee}$ and $\mathcal{Y}_{\mathbf{v}'}$, where $\mathbf{v}' \in \mathbf{e}$ is the vertex different from $\mathbf{v}$. One somehow has to make a Γ-invariant choice of which of the buildings $\mathbf{b}_1$ or $\mathbf{b}_2$ one wants to use to define these spaces. This might not be impossible, but it does make the construction somewhat more complicated. We will not follow this road.

9.6 Proposition. *Let $\mathbf{b}, \mathbf{b}' \in \mathcal{T}$ be two distinct $PU(2,L)$-buildings and let $\Sigma^\circ_{\mathbf{b}}, \Sigma^\circ_{\mathbf{b}'} \subset \mathcal{Y}^\circ$ be the associated subspaces. Then exactly one the following two statements hold:*

a) The intersection $\mathbf{b} \bigcap \mathbf{b}'$ is a vertex $\mathbf{v}$ and $\Sigma^\circ_{\mathbf{b}} \bigcap \Sigma^\circ_{\mathbf{b}'} = \mathcal{Y}_{\mathbf{v}} = \Sigma^\circ_{\mathbf{v},\mathbf{b}} = \Sigma^\circ_{\mathbf{v},\mathbf{b}'}$.

b) The intersection $\mathbf{b} \cap \mathbf{b}'$ is empty and the intersection $\Sigma^\circ_{\mathbf{b}} \bigcap \Sigma^\circ_{\mathbf{b}'} = \emptyset$.

Proof. Let $\mathcal{Y}_{\mathbf{v}'} \subset \mathcal{Y}^\circ$. One verifies that $\mathcal{Y}_{\mathbf{v}'} \bigcap \Sigma^\circ_{\mathbf{b}} \neq \emptyset$ if and only if $\mathbf{v}' \in \mathbf{b}$. From this and the definition of a transversal covering the proposition follows. $\square$

9.7. A map $\psi_{\mathcal{B}}^{\mathcal{Y}^\circ} : \mathcal{Y}^\circ \to \mathcal{B}$. We define a map $\psi_{\mathcal{B}}^{\mathcal{Y}^\circ} : \mathcal{Y}^\circ \to \mathcal{B}$ by taking:

$$\psi_{\mathcal{B}}^{\mathcal{Y}^\circ} = \begin{cases} \psi_{\mathcal{B}} & \text{for points contained in } \bigcup_{\mathbf{v} \in |\mathcal{T}|, \tau(\mathbf{v})=0} \mathcal{Y}_{\mathbf{v}} \\ \psi_{\mathcal{B}}^{\vee} & \text{for points contained in } \bigcup_{\mathbf{v} \in |\mathcal{T}|, \tau(\mathbf{v})=1} \mathcal{Y}_{\mathbf{v}}. \end{cases}$$

To avoid making the notation unnecessary complex, we do not distinguish between the use of the coordinates x_i and z_i, $i = 0,1,2$ for this map. In the propostion below we show that this is well-defined.

9.8 Proposition. *The following statements hold:*

i) The map $\psi_{\mathcal{B}}^{\mathcal{Y}^\circ}$ is well-defined and Γ-equivariant.

ii) $\Sigma^\circ_{\mathbf{b}} = \{x \in \mathcal{Y}^\circ \mid d_{\mathcal{B}}(\psi^{\mathcal{Y}^\circ}_{\mathcal{B}}(x), \mathbf{b}) < 1\} \subset \mathcal{Y}^\circ$

iii) If $x \in \mathcal{Y}^\circ$, then $d_{\mathcal{B}}(\psi^{\mathcal{Y}^\circ}_{\mathcal{B}}(x), |\mathcal{T}|) < 1$.

iv) $\psi^{\mathcal{Y}^\circ}_{\mathcal{B}}(\mathcal{Y}^\circ) = \{u \in \mathcal{B}(\mathbb{Q}) \mid d_{\mathcal{B}}(u, |\mathcal{T}|) < 1\} = \mathcal{B}(\mathbb{Q}) - \{\mathbf{v} \mid \mathbf{v} \notin \bigcup_{\mathbf{b}\in\mathcal{T}} \mathbf{b}\}$.

Proof. To show that the map $\psi^{\mathcal{Y}^\circ}_{\mathcal{B}}$ is well-defined we have to show that at edges $\mathbf{e} \in |\mathcal{T}|$ the equality $\psi_{\mathcal{B}}(x) = \psi^\vee_{\mathcal{B}}(z)$ holds, whenever $x \in \mathcal{Y}_{\mathbf{e}}$ is identified with $z \in \mathcal{Y}^\vee_{\mathbf{e}}$ in $\mathcal{Y}^\circ$. We observe that if $\mathbf{b} \in \mathcal{T}$ contains the edge $\mathbf{e}$, then $\psi_{\mathcal{B}}(x) = \psi_{\mathbf{b}}(x)$ and $\psi^\vee_{\mathcal{B}}(z) = \psi^\vee_{\mathbf{b}}(z)$. Since $g^*x_1g^*z_2 + g^*x_2g^*z_1 = 0$ holds and $\psi_{\mathbf{b}}(x) = v(\frac{g^*x_1}{g^*x_2})$ and $\psi^\vee_{\mathbf{b}}(z) = v(\frac{g^*z_1}{g^*z_2})$ after a suitable identification of the apartment $A \subset \mathbf{b}$, $A \ni \mathbf{e}$ with the real line $\mathbb{R}$, we conclude that $\psi^{\mathcal{Y}^\circ}_{\mathcal{B}}(x) = \psi_{\mathbf{b}}(x) = \psi^\vee_{\mathbf{b}}(z) = \psi^{\mathcal{Y}^\circ}_{\mathcal{B}}(z)$ holds.

The map $\psi^{\mathcal{Y}^\circ}_{\mathcal{B}}$ is Γ-equivariant by construction.

Statements (ii) and (iii) of the proposition follow directly from the fact that $d_{\mathcal{B}}(\psi_{\mathcal{B}}(x), \mathbf{v}) < 1$ for $x \in \mathcal{Y}_{\mathbf{v}}$. The final statement follows from the fact that $\mathcal{Y}_{\mathbf{e}} = \Sigma^\circ_{\mathbf{e},\mathbf{b}} = \Sigma_{\mathbf{e},\mathbf{b}}$. Therefore $\psi^{\mathcal{Y}^\circ}_{\mathcal{B}}(\mathcal{Y}_{\mathbf{e}}) = \psi_{\mathbf{b}}(\Sigma_{\mathbf{e},\mathbf{b}})$ and $\psi_{\mathbf{b}}(\Sigma_{\mathbf{e},\mathbf{b}}) = \mathbf{e} \bigcap \mathbb{Q} - \{\mathbf{v}_1, \mathbf{v}_2\}$. Here $\mathbf{v}_1$ and $\mathbf{v}_2$ are the vertices in $\mathbf{e}$. Clearly, $\mathbf{v} \in \psi^{\mathcal{Y}^\circ}_{\mathcal{B}}(\mathcal{Y}_{\mathbf{v}})$. Therefore the image of $\psi^{\mathcal{Y}^\circ}_{\mathcal{B}}$ contains all rational points of the building $\mathcal{B}$, except for the vertices $\mathbf{v} \in \mathcal{B} - |\mathcal{T}|$. □

10 Compactification

Let $\Gamma \subset PU(3, L)$ be a discrete co-compact subgroup preserving an almost complete transversal covering $\mathcal{T}$. Let $\mathcal{Y}^\circ$ be the analytic space corresponding to the covering $\mathcal{T}$ as constructed in §9. If the covering $\mathcal{T}$ is not complete, then the quotients $\mathcal{Y}^\circ/\Delta$ are not compact. Here $\Delta \subset \Gamma$ is a normal subgroup of finite index and without elements of finite order.

If the transversal covering $\mathcal{T}$ is Γ-adapted, then the space $\mathcal{Y}^\circ$ can be compactified Γ-invariantly. To compactify the quotient, one adds to the space $\mathcal{Y}^\circ$ a suitable admissable space $\mathcal{Y}_{\mathbf{v}}$ for each vertex $\mathbf{v} \in \mathcal{B}$ that is not contained in a $PU(2, L)$-building $\mathbf{b} \in \mathcal{T}$. The spaces $\mathcal{Y}_{\mathbf{v}}$ that are added are isomorphic to open admissable subsets of Ω_1.

The result is a space $\mathcal{Y} \supset \mathcal{Y}^\circ$ on which Γ acts discontinuously such that the quotients $\mathcal{Y}/\Delta$ are proper.

10.1. Analytic spaces for the vertices v and edges e not contained in $|\mathcal{T}|$. For each vertex $\mathbf{v} \in \mathcal{B}$ that is not contained in $|\mathcal{T}| = \bigcup_{\mathbf{b}\in\mathcal{T}} \mathbf{b}$, we define an analytic subspace $\mathcal{Y}_{\mathbf{v}} \subset Y^{s\vee}$. Let $g_{\mathbf{v}} \in SU(3,L)$ be an element such that the apartment $g_{\mathbf{v}}(A)$ contains the vertex $\mathbf{v}$. Let $g_{\mathbf{v}}^* z_0, g_{\mathbf{v}}^* z_1, g_{\mathbf{v}}^* z_2$ be the coordinates of $\mathbb{P}^\vee \cong \mathbb{P}^2_L$ such that the torus S belonging to the apartment $g_{\mathbf{v}}(A) \subset \mathcal{B}$ acts diagonally. The elements $g_{\mathbf{v}}$ are choosen in a Γ-invariant way.

We define $\mathcal{Y}_{\mathbf{v}} := \{z \in Y^{s\vee} \mid d_{\mathcal{B}}(\psi_{\mathcal{B}}^\vee(z), \mathbf{v}) < 1,\ g_{\mathbf{v}}^* z_0 = 0\}$. Then $\mathcal{Y}_{\mathbf{v}} \cong \{z \in \Omega_1 \mid d_{\mathbf{b}}(\psi_{\mathbf{b}}^\vee(z), \mathbf{v}) < 1\}$. Here the space $\Omega_1 \subset \mathbb{P}^1_L$, where the $\mathbb{P}^1_L \subset \mathbb{P}^\vee$ is given by $g_{\mathbf{v}}^* z_0 = 0$ and $\mathbf{b}$ is the $PU(2,L)$-building that contains the apartment $g_{\mathbf{v}}(A)$.

For an edge $\mathbf{e} \notin |\mathcal{T}|$ we define two spaces $\mathcal{Y}_{\mathbf{e}}$ and $\mathcal{Y}_{\mathbf{e}}^\vee$. Let $\mathbf{v}, \mathbf{v}' \in \mathbf{e}$ be the vertices of type $\tau(\mathbf{v}) = 1$ and $\tau(\mathbf{v}') = 0$. Let $\mathcal{Y}_{\mathbf{e}} := \{x \in \mathcal{Y}_{\mathbf{v}'} \mid \psi_{\mathcal{B}}(x) \in \mathbf{e}, \psi_{\mathcal{B}}(x) \neq \mathbf{v}'\} \subset Y^s$. Let $\mathcal{Y}_{\mathbf{e}}^\vee := \{z \in \mathcal{Y}_{\mathbf{v}} \mid \psi_{\mathcal{B}}(z) \in \mathbf{e}, \psi_{\mathcal{B}}^\vee(z) \neq \mathbf{v}\} \subset Y^{s\vee}$.

10.2 Lemma. *Let $\mathbf{e} \in \mathcal{B}$ be an edge that is not contained in $|\mathcal{T}| = \bigcup_{\mathbf{b} \in \mathcal{T}} \mathbf{b}$. Let $\mathbf{v}_0, \mathbf{v}_1 \in \mathbf{e}$ be the hyperspecial and non-hyperspecial vertex, respectively. Then $\mathcal{Y}_{\mathbf{e}} \cong \{z \in \mathcal{Y}_{\mathbf{e}}^\vee \mid d_{\mathcal{B}}(\psi_{\mathbf{b}}^\vee(z), \mathbf{v}_0) < \frac{1}{q+1}\}$. The isomorphism is given by taking $g_{\mathbf{v}}^* x_0 g_{\mathbf{v}}^* z_2 + g_{\mathbf{v}}^* x_2 g_{\mathbf{v}}^* z_1 = 0$, where the coordinates are such that $g_{\mathbf{v}}^* z_0 = 0$ defines the $\mathbb{P}^1_L \subset \mathbb{P}^\vee$ used to define $\mathcal{Y}_{\mathbf{v}_1} \subset Y^{s\vee}$ for the vertex $\mathbf{v}_1 \in \mathbf{e}$.*

Proof. Without loss of generality, we may assume that $g_{\mathbf{v}_1} = 1$. Then we may assume the points $x \in \mathcal{Y}_{\mathbf{v}_0}$ satisfy the equation $b_{\mathbf{v}_0}(x) = x_1 x_2^q + x_1^q x_2 + x_0^{q+1} = 0$. The isotropic point in $Iso(\mathbf{v}_0, \mathbf{b})$ that corresponds to the edge $\mathbf{e}$ is assumed to be $(0, 0, \overline{x_2})$.

The points $x \in \mathcal{Y}_{\mathbf{e}}$ satisfy $\psi_{\mathcal{B}}(x) \in \mathbf{e}$. Therefore $1 > |\frac{x_1}{x_2}| > |\pi|$ holds. From $b_{\mathbf{v}_0}(x) = 0$, it follows that $\frac{x_1}{x_2} \equiv -\frac{x_0^{q+1}}{x_2^{q+1}} \mod \pi$ holds. Therefore one obtains a $\mathbb{P}^1_\ell$ in the reduction with coordinates x_0, x_2 (and not x_1, x_2). The vertex $\mathbf{v}_1 \in \mathbf{e}$ satisfies $|\frac{x_1}{x_2}| = |\pi|$. The L°-module corresponding to the vertex $\mathbf{v}_1$ is $\langle \pi^{\frac{1}{q+1}} \cdot e_0, \pi \cdot e_1, e_2 \rangle$. This gives indeed a line $\mathbb{P}^1_\ell$ with coordinates x_0 and x_2. Moreover, the points $x \in \mathcal{Y}_{\mathbf{e}}$ satisfy $1 > |\frac{x_0}{x_1}| > |\frac{1}{q+1}|$.

The identification of $(x_0, x_1, x_2) \in \mathcal{Y}_{\mathbf{e}}$ with $(0, z_1, z_2) \in \mathcal{Y}_{\mathbf{e}}^\vee$ using the relation $x_0 z_2 + x_2 z_1 = 0$ gives an isomorphism $\mathcal{Y}_{\mathbf{e}} \cong \{z \in \mathcal{Y}_{\mathbf{e}}^\vee \mid d_{\mathcal{B}}(\psi_{\mathbf{b}}^\vee(z), \mathbf{v}_0) < \frac{1}{q+1}\}$ as stated in the lemma. □

10.3 Lemma. *Let $\mathbf{v} \in \mathcal{B} - |\mathcal{T}|$ be a (non-hyperspecial) vertex. Let $\Gamma_{\mathbf{v}} \subset \Gamma$ be the stabilizer of the vertex. Then the following holds:*

i) There exists an anisotropic point $x_0 \in \mathbb{P}^2_L$ fixed by $\Gamma_{\mathbf{v}}$. The restriction $\Gamma_{\mathbf{v}}|_{x_0^\perp}$ defines an embedding $\Gamma_{\mathbf{v}} \hookrightarrow PU(2,L)$.

ii) There exists a $\Gamma_{\mathbf{v}}$-invariant pairing of edges $\mathbf{e} \ni \mathbf{v}_1$.

Proof. The group $\Gamma_{\mathbf{v}}$ is contained in the stabiliser $P_{\mathbf{v}} \subset \Gamma \subset PU(3, L)$. Since the vertex $\mathbf{v}$ is non-hyperspecial, it follows that $\Gamma_{\mathbf{v}}$ stabilizes some anisotropic point x_0. The transversal covering $\mathcal{T}$ is Γ-adapted. In particular, no element of $\Gamma_{\mathbf{v}}$ stabilises an edge $\mathbf{e} \ni \mathbf{v}$. From this it follows that the restriction $\Gamma_{\mathbf{v}}|_{x_0^\perp} \cong \Gamma_{\mathbf{v}}$. From this statement (i) follows.

Let us now consider statement (ii). From $p > 2$, it follows that the number of edges $\mathbf{e} \ni \mathbf{v}$ is even. Since $\mathcal{T}$ is Γ-adapted, the stabiliser $\Gamma_{\mathbf{e}} \subset \Gamma_{\mathbf{v}}$ of an edge $\mathbf{e} \ni \mathbf{v}$ is trivial. If the number of $\Gamma_{\mathbf{v}}$ orbits on edges $\mathbf{e} \ni \mathbf{v}$ is even, then we can choose a $\Gamma_{\mathbf{v}}$-invariant pairing $\{\gamma'(\mathbf{e}), \gamma'(\mathbf{e}')\}$, $\gamma' \in \Gamma_{\mathbf{v}}$, such that the edges $\mathbf{e}, \mathbf{e}' \ni \mathbf{v}$ are in two different $\Gamma_{\mathbf{v}}$-orbits. Since there are an even number of $\Gamma_{\mathbf{v}}$-orbits, we can obtain a pairing by repeating this process for other orbits.

If the number of orbits is odd, then the order $|\Gamma_{\mathbf{v}}|$ is even. We choose an element $\gamma \in \Gamma_{\mathbf{v}}$ of order two. In a $\Gamma_{\mathbf{v}}$-orbit of edges $\mathbf{e} \ni \mathbf{v}$ we choose an edge $\mathbf{e}_0$. Then the pairing on the orbit is given by $\{\gamma'(\mathbf{e}_0), \gamma'(\gamma(\mathbf{e}_0))\}$. Since $\gamma'(\gamma(\mathbf{e}_0)) \neq \mathbf{e}_0$ for all $\gamma' \in \Gamma_{\mathbf{v}}$ and $\gamma'(\gamma(\mathbf{e}_0)) = \mathbf{e}_0$ implies $\gamma' = \gamma$, this gives a well-defined $\Gamma_{\mathbf{v}}$-invariant pairing on the orbit. Repeating the process for each $\Gamma_{\mathbf{v}}$ orbit of edges $\mathbf{e} \ni \mathbf{v}$ gives the required pairing of edges. □

10.4 Definition. We are now able to glue all the spaces $\mathcal{Y}_{\mathbf{v}}$, $\mathbf{v} \in \mathcal{B}$ together Γ-equivariantly along the spaces $\mathcal{Y}_{\mathbf{e}}$ and $\mathcal{Y}_{\mathbf{e}}^\vee$. For each vertex $\mathbf{v} \in \mathcal{B} - |\mathcal{T}|$ we fix a $\Gamma_{\mathbf{v}}$-equivariant pairing of the edges $\mathbf{e} \ni \mathbf{v}$. We choose these pairing Γ-invariant. Each such pairing induces a pairing of coordinates x_1 and x_2 that is used in lemma 10.2 above to obtain an isomorphism $\mathcal{Y}_{\mathbf{e}} \cong \{z \in \mathcal{Y}_{\mathbf{v}}^\vee | d_{\mathcal{B}}(\psi_{\mathbf{b}}^\vee(z), \mathbf{v}_0) < \frac{1}{q+1}\}$. The analytical space $\mathcal{Y} := \bigcup_{\mathbf{v} \in \mathcal{B}} \mathcal{Y}_{\mathbf{v}} / \sim$, is defined by the equivalence relation given by the following identifications;

$$\begin{cases} \mathcal{Y}_{\mathbf{e}} \cong \mathcal{Y}_{\mathbf{e}}^\vee & \text{if } \mathbf{e} \in |\mathcal{T}| \\ \mathcal{Y}_{\mathbf{e}} \cong \{z \in \mathcal{Y}_{\mathbf{e}}^\vee \mid d_{\mathcal{B}}(\psi_{\mathbf{b}}^\vee(z), \mathbf{v}_0) < \frac{1}{q+1}\} & \text{if } \mathbf{e} \notin |\mathcal{T}|. \end{cases}$$

10.5 Theorem. *Let $\Gamma \subset PU(3, L)$ be a discrete co-compact subgroup and let $\Delta \subset \Gamma$ be a subgroup of finite index that contains no elements of finite order. Let $\mathcal{T}$ be a Γ-adapted transversal covering of $\mathcal{B}$ and let $\mathcal{Y} := \bigcup_{\mathbf{v} \in \mathcal{B}} \mathcal{Y}_{\mathbf{v}} / \sim$ be the analytical space corresponding to the covering $\mathcal{T}$. Then the following statements hold:*

i) The space $\mathcal{Y}$ is a well-defined rigid analytic variety.

ii) The group Γ acts discretely on the rigid analytical space $\mathcal{Y}$ and the quotient $\mathcal{Y}/\Delta$ is a proper algebraic curve on which the finite group Γ/Δ acts.

Proof. It is clear from the construction that $\mathcal{Y}$ is a well-defined rigid analytic space. The group Δ acts on $\mathcal{Y}$ by permuting the admissable subspaces $\mathcal{Y}_{\mathbf{v}} \subset \mathcal{Y}$. Since Γ and Δ act discretely on the building, they act discretely on the space $\mathcal{Y}$.

Let us now prove that the quotient $\mathcal{Y}/\Delta$ is proper. To do this we cover $\mathcal{Y}$ with two admissable Γ-invariant affinoid coverings $\{X_{\mathbf{v}}(r_1) \mid \mathbf{v} \in \mathcal{B}\}$ and $\{X_{\mathbf{v}}(r_2) \mid \mathbf{v} \in \mathcal{B}\}$ such that $X_{\mathbf{v}}(r_1) \subset\subset X_{\mathbf{v}}(r_2)$ for all vertices $\mathbf{v} \in \mathcal{B}$.

Let $X_{\mathbf{v}}(R) \subset \mathcal{Y}_{\mathbf{v}}$ be the analytical subspace $X_{\mathbf{v}}(R) := \{x \in \mathcal{Y}_{\mathbf{v}} \mid d_{\mathcal{B}}(\psi_{\mathcal{B}}(x), \mathbf{v}) \leq R\}$ if $\tau(\mathbf{v}) = 0$ and $X_{\mathbf{v}}(R) := \{z \in \mathcal{Y}_{\mathbf{v}} \mid d_{\mathcal{B}}(\psi^{\vee}_{\mathcal{B}}(z), \mathbf{v}) \leq R\}$ if $\tau(\mathbf{v}) = 1$. Here $R \in \mathbb{Q}$ is such that $0 \leq R < 1$. One verifies that the analytical space $X_{\mathbf{v}}(R)$ is in fact an affinoid subspace of $\mathcal{Y}_{\mathbf{v}}$. If $0 \leq R < R' < 1$, then $X_{\mathbf{v}}(R) \subset\subset X_{\mathbf{v}}(R')$ holds. If $1 > R \geq \frac{q+1}{q+2}$, then the union $\bigcup_{\mathbf{v}\in\mathcal{B}} X_{\mathbf{v}}(R) = \mathcal{Y}$. The condition $R \geq \frac{q+1}{q+2}$ is needed, because for edges $\mathbf{e} \in \mathcal{B} - |\mathcal{T}|$ the identification of a point $x \in \mathcal{Y}_{\mathbf{e}}$ with a point $z \in \mathcal{Y}^{\vee}_{\mathbf{e}}$ is such that $d_{\mathcal{B}}(\psi^{\vee}_{\mathbf{b}}(z), \mathbf{v}_0) = \frac{1}{q+1} \cdot d_{\mathcal{B}}(\psi_{\mathbf{b}}(x), \mathbf{v}_0)$. Here $\mathbf{v}_0 \in \mathbf{e}$ is the hyperspecial vertex. By construction the covering is Γ-invariant. Therefore we can take r_1 and r_2 such that $\frac{q+1}{q+2} \leq r_1 < r_2 < 1$ holds to obtain our admissable coverings.

After taking, if necessary, a suitable subgroup of Δ of finite index, we may assume that the action of Γ on the affinoids is such that $\gamma(X_{\mathbf{v}}(r_i)) \bigcap X_{\mathbf{v}}(r_i) = \emptyset$, $i = 1, 2$ holds for all $\gamma \in \Gamma$, $\gamma \neq 1$ and for all $\mathbf{v} \in \mathcal{B}$. Then the quotient $\mathcal{Y}/\Gamma$ is covered by the image of finitely many affinoids $X_{\mathbf{v}_i}(r_1)$ and $X_{\mathbf{v}_i}(r_2)$ with $i = 1, \ldots, s$ for some $s \geq 1$. From this the properness of the quotient follows. Since $\mathcal{Y}/\Delta$ is a curve it is algebraic. □

10.6 Corollary. *The following two statements hold:*

i) The group $\Gamma \subset PU(3, L)$ acts linearly on $\bigcup_{\mathbf{v}\in\mathcal{B},\tau(\mathbf{v})=0} \mathcal{Y}_{\mathbf{v}}$ through the coordinates x_i, $i = 0, 1, 2$.

ii) The group $\Gamma \subset PU(3, L)$ acts linearly on $\bigcup_{\mathbf{v}\in\mathcal{B},\tau(\mathbf{v})=1} \mathcal{Y}_{\mathbf{v}}$ through the coordinates z_i, $i = 0, 1, 2$.

Proof. Clear from the construction. □

10.7. Construction for subtrees $T \subset \mathcal{B}$.

The construction can easily be generalised to the case of a subtree T of the building $\mathcal{B}$. Let $\Gamma \subset PU(3, L)$ be a discrete subgroup acting on a subtree $T \subset \mathcal{B}$ with a finite quotient T/Γ. Let $\mathcal{T}_T$ be a Γ-invariant almost complete transversal covering of T by $PU(2, L)$-buildings. Then all hyperspecial vertices $\mathbf{v} \in \mathcal{T}$ are contained in $|\mathcal{T}_T|$ and the buildings $\mathbf{b} \in \mathcal{T}_T$ intersect transversally.

One first constructs a uniformising space $\mathcal{Y}_T^\circ$ by glueing admissable subspaces $\Sigma_{\mathbf{b},T}^\circ \subset \Sigma_{\mathbf{b}}$ with $\mathbf{b} \in \mathcal{T}_T$. The subspaces $\Sigma_{\mathbf{b},T}^\circ \subset \Sigma_{\mathbf{b}}$ are obtained by removing the balls $B(a, |\pi|)$ for $a \in Iso(\mathbf{v}, \mathbf{b})_T$ with $\mathbf{v} \in \mathbf{b}$. Here $Iso(\mathbf{v}, \mathbf{b})_T \subset Iso(\mathbf{v}, \mathbf{b})$ consists of the isotropic points that correspond to an edge $\mathbf{e} \in T$, $\mathbf{v} \in \mathbf{e}$, that is not contained in $\mathbf{b}$.

If the transversal covering is Γ-admissable, then the space $\mathcal{Y}_T^\circ$ can be compactified into an analytic variety $\mathcal{Y}_T$ on which the group Γ acts discontinuously.

10.8 Example. Let $\Gamma \subset PU(3, L)$ be a discrete co-compact subgroup and $\mathcal{T}$ a Γ-invariant transversal covering. Let us assume that $\mathcal{T}$ is not complete and that the group Γ acts transitively on the connected components of $|\mathcal{T}| = \cup\{\mathbf{b} \in \mathcal{T}\} \subset \mathcal{B}$. Let $T = |\mathcal{T}|^\circ \subset |\mathcal{T}|$ be a connected component and let $\Gamma_T \subset \Gamma$ be the stabiliser of T. Then the quotient T/Γ_T is finite.

The transversal covering $\mathcal{T}_T$ of T is complete and therefore Γ_T-adapted. Let us assume that the transversal covering $\mathcal{T}$ is Γ-adapted. In particular, we can construct rigid varieties $\mathcal{Y}$ and $\mathcal{Y}_T$ on which the groups Γ and Γ_T act discontinuously, respectively. Let $\Delta \subset \Gamma$ be a subgroup of finite index without elements of finite order. We assume that Δ acts transitively on the connected components of $|\mathcal{T}|$. Let $\Delta_T \subset \Delta$ be the stabiliser of the tree T.

The curves $\mathcal{Y}/\Delta$ and $\mathcal{Y}_T^\circ/\Delta = \mathcal{Y}_T/\Delta_T$ both contain the open admissable subspace $\mathcal{Y}^\circ/\Delta$. They are different compactifications of the space $\mathcal{Y}^\circ/\Delta$. Let $\varphi_T : \mathcal{Y}_T \longrightarrow \mathcal{Y}_T/\Delta_T$ be the quotient map. Let $\mathcal{Y}_T^{\circ\circ} \subset \mathcal{Y}_T$ be the open subspace $\varphi_T^{-1}(\mathcal{Y}^\circ/\Delta)$. Then $\mathcal{Y}^\circ/\Delta = \mathcal{Y}_T^{\circ\circ}/\Delta_T$ holds. Since $\Delta_T \subset \Delta$ is not normal, the quotient map $\mathcal{Y}^\circ \longrightarrow \mathcal{Y}^\circ/\Delta$ does not factor through the space $\mathcal{Y}_T^{\circ\circ}$.

11 Reduction and genus

We define a Γ-invariant map from the space $\mathcal{Y}$ to the building $\mathcal{B}$. Using this map we construct a Γ-invariant pure affinoid covering of $\mathcal{Y}$. A description of the reduction of $\mathcal{Y}$ w.r.t. this covering is given. We also describe a semistable reduction and give a formula for the genus of the quotients $\mathcal{Y}/\Delta$ in terms of its reduction.

11.1 Definition. *A map* $\psi_{\mathcal{B}}^{\mathcal{Y}}: \mathcal{Y} \to \mathcal{B}$. Let us define a map $\psi_{\mathcal{B}}^{\mathcal{Y}}: \mathcal{Y} \to \mathcal{B}$ by:

$$\psi_{\mathcal{B}}^{\mathcal{Y}} = \begin{cases} \psi_{\mathcal{B}} & \text{on } \bigcup_{\mathbf{v}\in\mathcal{B},\tau(\mathbf{v})=0} \mathcal{Y}_{\mathbf{v}} - \bigcup_{\mathbf{e}\in\mathcal{B}-|\mathcal{T}|} \mathcal{Y}_{\mathbf{e}} \\ \psi_{\mathcal{B}}^{\vee} & \text{on } \bigcup_{\mathbf{v}\in\mathcal{B},\tau(\mathbf{v})=1} \mathcal{Y}_{\mathbf{v}}. \end{cases}$$

We again do not distinguish between the coordinates x_i and z_i, $i = 0, 1, 2$ for the map $\psi_{\mathcal{B}}^{\mathcal{Y}}$. In the proposition below we show that the map is well-defined.

11.2 Proposition. *The following statements hold:*

i) The map $\psi_{\mathcal{B}}^{\mathcal{Y}}: \mathcal{Y} \to \mathcal{B}$ *is well-defined and* Γ*-equivariant.*

ii) $\psi_{\mathcal{B}}^{\mathcal{Y}}|_{\mathcal{Y}^\circ - \bigcup_{\mathbf{e}\in\mathcal{B}-|\mathcal{T}|}\mathcal{Y}_{\mathbf{e}}} = \psi_{\mathcal{B}}^{\mathcal{Y}^\circ}$.

iii) The complement of $\mathcal{Y}^\circ$ *in* $\mathcal{Y}$ *is* $\mathcal{Y}-\mathcal{Y}^\circ = \{x \in \mathcal{Y} \mid d_{\mathcal{B}}(\psi_{\mathcal{B}}^{\mathcal{Y}}(x), |\mathcal{T}|) \geq \frac{1}{q+1}\}$ $= \{x \in \mathcal{Y} \mid \exists(\mathbf{v} \in \mathcal{B} - |\mathcal{T}|)\ d_{\mathcal{B}}(\psi_{\mathcal{B}}^{\mathcal{Y}}(x), \mathbf{v}) \leq \frac{q}{q+1}\}$.

Proof. Clear from the definition. □

11.3 Definition. *Affinoid subspaces of the space* $\mathcal{Y}$. For each edge $\mathbf{e} \in \mathcal{B}$ and each vertex $\mathbf{v} \in \mathcal{B}$ affinoid subspaces $X_{\mathbf{e}}^{\mathcal{Y}}, X_{\mathbf{v}}^{\mathcal{Y}} \subset \mathcal{Y}$ are obtained by taking $X_{\mathbf{v}}^{\mathcal{Y}} := \{x \in \mathcal{Y} \mid \psi_{\mathcal{B}}^{\mathcal{Y}}(x) = \mathbf{v}\}$ and $X_{\mathbf{e}}^{\mathcal{Y}} := \{x \in \mathcal{Y} \mid \psi_{\mathcal{B}}^{\mathcal{Y}}(x) \in \mathbf{e}\}$. Let $\mathcal{C}$ be the covering $\mathcal{C} := \{X_{\mathbf{v}}^{\mathcal{Y}}, X_{\mathbf{e}}^{\mathcal{Y}} \mid \mathbf{v}, \mathbf{e} \in \mathcal{B}\}$ of the space $\mathcal{Y}$.

The affinoid spaces $X_{\mathbf{e}}^{\mathcal{Y}}$ are obtained by removing some points from the reduction of the spaces $X_{\mathbf{e}}^{\Sigma}$ for $\mathbf{e} \in |\mathcal{T}|$. Therefore the affinoid algebra is not uniquely determined by our definition. Over the field L not both h and $\frac{\pi^{\frac{1}{q+1}}}{h}$ are defined. As in the case of Σ we choose the affinoid algebra in such a way that the components at the hyperspecial vertices have multiplicity $q+1$ and the components at the other vertices have multiplicity one.

11.4 Theorem. *The covering* $\mathcal{C} = \{X_{\mathbf{v}}^{\mathcal{Y}}, X_{\mathbf{e}}^{\mathcal{Y}} \mid \mathbf{v}, \mathbf{e} \in \mathcal{B}\}$ *is a* Γ*-invariant pure affinoid covering of the rigid analytic space* $\mathcal{Y}$*. The reduction of the space* $\mathcal{Y}$ *w.r.t.* $\mathcal{C}$ *is as follows:*

i) To each vertex $\mathbf{v} \in |\mathcal{T}| \subset \mathcal{B}$ *corresponds a hermitian curve* $\mathcal{H}$*. If the vertex* $\mathbf{v}$ *is hyperspecial, then this component has multiplicity* $q+1$*, otherwise it has multiplicity one.*

ii) To each vertex $\mathbf{v} \in \mathcal{B} - |\mathcal{T}|$ corresponds a projective line $\mathbb{P}^1_\ell$ with on it a hermitian form.

iii) The components of the reduction corresponding to the vertices $\mathbf{v}, \mathbf{v}' \in \mathcal{B}$ intersect if and only if there exists an edge $\mathbf{e} \ni \mathcal{B}$ with $\mathbf{v}, \mathbf{v}' \in \mathbf{e}$. In that case, the point of intersection is ℓ-valued (and isotropic) in both.

Proof. It follows directly from the definition that the covering $\mathcal{C}$ is pure. From the definitions of the affinoids one easily obtains a description of the affinoids and their reduction. This is similar to the determination of the reduction of Σ in theorem 5.8. We leave the details to the reader. $\square$

11.5 Remark. *Alternative pure affinoid coverings of the space $\mathcal{Y}$.* One can construct a pure affinoid covering of $\mathcal{Y}$ by affinoids $X_\mathbf{v}(R)$ by using two distinct values R_1 and R_2. Let us take for vertices $\mathbf{v} \in |\mathcal{T}|$ the affinoids $X_\mathbf{v}(R_1)$ and for the vertices $\mathbf{v} \in \mathcal{B} - |\mathcal{T}|$ the affinoids $X_\mathbf{v}(R_2)$ with $0 < R_1, R_2 < 1$. The covering is an admissable affinoid covering of $\mathcal{Y}$ if $\frac{1}{2} \leq R_1 < 1$ and $\frac{2q+1}{2(q+1)} \leq R_2 < 1$. The covering is pure if $R_1 = \frac{1}{2}$ and $R_2 = \frac{2q+1}{2(q+1)}$.

Let us define affinoid spaces $X^\mathcal{Y}_\mathbf{v}(R)$ for vertices $\mathbf{v} \in \mathcal{B}$ as follows: $X^\mathcal{Y}_\mathbf{v}(R) := \{x \in \mathcal{Y} \mid d_\mathcal{B}(\psi^\mathcal{Y}_\mathcal{B}(x), \mathbf{v}) \leq R\}$ with $0 < R < 1$. For $\frac{1}{2} \leq R < 1$ the covering is an admissable affinoid covering of the space $\mathcal{Y}$. If $R = \frac{1}{2}$, then the covering is a pure affinoid covering of $\mathcal{Y}$.

For hyperspecial vertices $\mathbf{v}$ such that all edges $\mathbf{e} \ni \mathbf{v}$ are contained in $|\mathcal{T}|$ and for non-hyperspecial vertices $\mathbf{v}$ the equality $X^\mathcal{Y}_\mathbf{v}(R) = X_\mathbf{v}(R)$ holds.

11.6 Definition. Let $\mathbf{e} \in \mathcal{B}$ be an edge and let $\mathbf{v}_0 \in \mathbf{e}$ be the hyperspecial vertex. We identify the edge $\mathbf{e}$ with the interval $[0,1] \subset \mathbb{R}$ such that the hyperspecial vertex corresponds with 0. We subdivide the edge $\mathbf{e}$ into q subintervals $\mathbf{e}(i) := [\frac{i}{q+1}, \frac{i+1}{q+1}]$, $i = 0, \ldots, q$. Let $\mathbf{v}_\mathbf{e}(i)$, $i = 0, \ldots, q+1$ denote the vertex corresponding to $\frac{i}{q+1}$. Then $\mathbf{v}_\mathbf{e}(0) := \mathbf{v}_0$ is the hyperspecial vertex and $\mathbf{v}_\mathbf{e}(q+1)$ non-hyperspecial vertex of $\mathbf{e}$.

The building with the edges $\mathbf{e} \notin |\mathcal{T}|$ thus subdivided will be denoted by $\mathcal{B}_\mathcal{T}$. More precisely $\{\mathbf{v} \in \mathcal{B}_\mathcal{T}\} = \{\mathbf{v} \in \mathcal{B}\} \cup \{\mathbf{v}_\mathbf{e}(i) \mid \mathbf{e} \in \mathcal{B} - |\mathcal{T}|, i = 1, \ldots, q\}$ and $\{\mathbf{e} \in \mathcal{B}_\mathcal{T}\} = \{\mathbf{e} \in |\mathcal{T}|\} \cup \{\mathbf{e}(i) \mid \mathbf{e} \in \mathcal{B} - |\mathcal{T}|, i = 0, \ldots, q\}$.

Let $\mathcal{C}_\mathcal{T}$ be the pure affinoid covering $\mathcal{C}_\mathcal{T} := \{X^\mathcal{Y}_\mathbf{v}, X^\mathcal{Y}_\mathbf{e} \mid \mathbf{v}, \mathbf{e} \in \mathcal{B}_\mathcal{T}\}$. Here $X^\mathcal{Y}_\mathbf{v} := \{x \in \mathcal{Y} \mid \psi^\mathcal{Y}_\mathcal{B}(x) \in \mathbf{v}\}$ and $X^\mathcal{Y}_\mathbf{e} := \{x \in \mathcal{Y} \mid \psi^\mathcal{Y}_\mathcal{B}(x) \in \mathbf{e}\}$ are defined as above, but with edges and vertices in $\mathcal{B}_\mathcal{T}$ instead of $\mathcal{B}$.

11.7 Proposition. *The affinoid covering $\mathcal{C}_\mathcal{T}$ is a pure affinoid covering of $\mathcal{Y}$. The reduction of $\mathcal{Y}$ w.r.t. to the covering $\mathcal{C}_\mathcal{T}$ is as follows:*

i) A component $\mathcal{H}$ for the vertices $\mathbf{v} \in |\mathcal{T}|$ and a component $\mathbb{P}^1_\ell$ with on it a hermitian form for vertices $\mathbf{v} \in \mathcal{B}_\mathcal{T} - |\mathcal{T}|$

ii) Two components intersect in an ℓ-valued isotropic point if and only if the corresponding vertices that form an edge in $\mathcal{B}_\mathcal{T}$.

iii) Over $L(\pi^{\frac{1}{q+1}})$ the reduction is semistable.

Proof. The description of the reduction follows directly from the definitions. The extension to $L(\pi^{\frac{1}{q+1}})$ is needed to ensure that all components $\mathcal{H}$ have multiplicity one. For edges $\mathbf{e} \notin |\mathcal{T}|$ the affinoid is essentially given by $f \cdot \frac{\pi}{f} = \pi$ for some suitable element f in the affinoid algebra belonging to $X^{\mathcal{Y}}_{\mathbf{e}}$. To obtain a divisor with normal crossings one needs to subdivide this affinoid as is done in definition 11.6. □

11.8 Proposition. *The genus of a curve $\mathcal{Y}/\Delta$ is given by $g(Y/\Delta) = 1 + n_\mathbf{e} + \sum_{\mathbf{v} \in \mathcal{B}/\Delta}(g_\mathbf{v} - 1) = 1 + n_\mathbf{e} + \frac{q^2-q-2}{2}\sharp\{\mathbf{v} \in |\mathcal{T}|/\Delta\} - \sharp\{\mathbf{v} \in (\mathcal{B} - |\mathcal{T}|)/\Delta\}$.*

Proof. The genus of a curve and the genus of its reduction are the same (See e.g. [F-P] proposition 5.6.2). Therefore we can use the description of the reduction to compute the genus of the curve $\mathcal{Y}/\Delta$. The genus of the hermitian curve $\mathcal{H}$ equals $\frac{q^2-q}{2}$. To the vertices $\mathbf{v} \in |\mathcal{T}|$ corresponds a hermitian curve and to the vertices $\mathbf{v} \in \mathcal{B} - |\mathcal{T}|$ a line $\mathbb{P}^1$. From this the formula follows. □

12 Comparison and speculation

We show that the spaces $\mathcal{Y}$ corresponding to complete transversal coverings differ from some known moduli spaces. We further speculate about systems of étale coverings of $\mathcal{Y}$ and on how the construction presented here can be generalised to other groups.

12.1 Definition. Let $\mathcal{T}_1, \mathcal{T}_2$ be two distinct complete transversal coverings of the building $\mathcal{B}$, i.e. $|\mathcal{T}_1| = |\mathcal{T}_2| = \mathcal{B}$. We call $\mathcal{T}_1$ and $\mathcal{T}_2$ *isomorphic* and write $\mathcal{T}_1 \cong \mathcal{T}_2$ if there exists an element $g \in PU(3, L)$ such that $g(\mathcal{T}_1) = \mathcal{T}_2$.

12.2 Lemma. *Let $\mathcal{T}_1 \ncong \mathcal{T}_2$ be two complete transversal coverings of $\mathcal{B}$. Let $\mathcal{Y}_1$ and $\mathcal{Y}_2$ be the uniformizing spaces belonging to $\mathcal{T}_1$ and $\mathcal{T}_2$, respectively. Then the generic fibres of $\mathcal{Y}_1$ and $\mathcal{Y}_2$ are isomorphic as point sets, but not as rigid analytic spaces.*

Proof. To proof that the sets of points in the generic fibres of $\mathcal{Y}_1$ and $\mathcal{Y}_2$ are isomorphic we construct isomorphic non-admissable coverings of both $\mathcal{Y}_1$ and $\mathcal{Y}_2$. For hyperspecial vertices $\mathbf{v}_0 \in \mathcal{B}$ we define $X_{\mathbf{v}_0}(i) := \{x \in \mathcal{Y}_i \mid \psi_{\mathcal{B}}^{\mathcal{Y}_i}(x) = \mathbf{v}_0\}$ for $i = 1, 2$. For non-hyperspecial vertices $\mathbf{v}_1 \in \mathcal{B}$ we define $Y_{\mathbf{v}_1}(i) := \{x \in \mathcal{Y}_i \mid d_{\mathcal{B}}(\psi_{\mathcal{B}}^{\mathcal{Y}_i}(x), \mathbf{v}_1) < 1\}$ for $i = 1, 2$.

The covering $\mathcal{C}_i := \{X_{\mathbf{v}_0}(i), Y_{\mathbf{v}_1}(i) \mid \mathbf{v}_0, \mathbf{v}_1 \in \mathcal{B}, \tau(\mathbf{v}_0) = 0, \tau(\mathbf{v}_1) = 1\}$ cover the space $\mathcal{Y}_i$ for $i = 1, 2$. One easily sees that $X_{\mathbf{v}_0}(1) \cong X_{\mathbf{v}_0}(2)$ and $Y_{\mathbf{v}_1}(1) \cong Y_{\mathbf{v}_1}(2)$ hold. Moreover, each point in the generic fibre of the space $\mathcal{Y}_i$ is contained in exactly one of the analytic spaces contained in the covering $\mathcal{C}_i$, $i = 1, 2$. Therefor the generic fibres of $\mathcal{Y}_1$ and $\mathcal{Y}_2$ are isomorphic as point sets. None of the spaces contained in the covering $\mathcal{Y}_i$, $i = 1, 2$, intersect. Hence these coverings are not admissable.

To show that the spaces $\mathcal{Y}_1$ and $\mathcal{Y}_2$ are not isomorphic as analytic spaces requires more work. TO DO!! $\square$

12.3 Comparison. *No relation with certain known moduli spaces.* In [Vol] a moduli space of principally polarized abelian varieties of dimension three related to the group $GU(1, 2)$ over $\mathbb{Q}_p$ is described in detail. The components of the reduction of the supersingular locus are hermitian curves $\mathcal{H}$ and correspond to the hyperspecial vertices of the building. The $p + 1$ components of the reduction belonging to the hyperspecial vertices that are neighbours of a single non-hyperspecial vertex $\mathbf{v}_1$ intersect in a single $\mathbb{F}_{p^2}$-valued point. (See corollary 6.2 and theorem 4 of [Vol].)

Obviously, this closed fibre differs from the closed fibres of the spaces $\mathcal{Y}$ for a complete transversal covering $\mathcal{T}$ of $\mathcal{B}$. In particular, the spaces $\mathcal{Y}$ seem to be unrelated to this moduli space.

In [Vol-W] example 4.8 a similar moduli space for the group $GU(1, 3)$ is described. The reduction is again a tree of Hermitian curves that intersect in the $\mathbb{F}_{p^2}$-valued points. In this case each such point is contained in $p^3 + 1$ curves. Again the reduction is far removed from that of our spaces $\mathcal{Y}$.

12.4 Speculation. *System of étale coverings of $\mathcal{Y}$.* The space Σ is a connected component of a space that is part of a system of $SU(2, L)$-equivariant étale coverings of Ω_1. Therefore the space Σ itself has a system of $SU(2, L)$-equivariant étale coverings. Each element of this system of coverings is again a connected component of an étale covering of Ω_1.

Let $\Gamma \subset PU(3, L)$ be a discrete co-compact subgroup that admits a complete transversal covering $\mathcal{T}$. Let $\mathcal{Y} = \mathcal{Y}^\circ = \bigcup_{\mathbf{b} \in \mathcal{T}} \Sigma_{\mathbf{b}}^\circ / \sim_{\mathcal{T}}$ be the analytic space that corresponds to the covering $\mathcal{T}$.

Each space $\Sigma_{\mathbf{b}}^\circ$, $\mathbf{b} \in \mathcal{T}$ is an open admissable subspace of a space Σ. The group $SU(2, L)$ belonging to $\mathbf{b}$ acts on $\Sigma_{\mathbf{b}}^\circ$. In particular, each space $\Sigma_{\mathbf{b}}^\circ$ admits a system of $SU(2, L)$-equivariant étale coverings. It seems likely that these systems for the spaces $\Sigma_{\mathbf{b}}^\circ$, $\mathbf{b} \in \mathcal{T}$ can be glued together into a system of Γ-equivariant étale coverings of the space $\mathcal{Y}$. The Galois group of such a system of étale coverings would be the stabiliser of Σ in the system of étale coverings of Ω_1.

Let us describe the Galois group in some detail. Let $D \supset K$ be a central division algebra with invariant 1/2 and maximal order $\mathcal{O}_D \subset D$. Then $D = L[\Pi]/(\Pi^2 - \pi)$ and $\mathcal{O}_D = L^\circ[\Pi]/(\Pi^2 - \pi)$. The element $\Pi \in D$ acts on elements $a \in L$ as $\Pi a = \sigma(a)\Pi$, where σ is the generator of $Gal(L/K)$. The Galois group of this system of étale coverings of Ω_1 is the group $\mathcal{O}_D^*$.

The n-th level of this system is an étale covering of Ω_1 with Galois group $(\mathcal{O}_D/\Pi^n\mathcal{O}_D)^*$. Drinfel'd only considers the levels $2m$, $m \in \mathbb{Z}$ that have Galois groups $(\mathcal{O}_D/\pi^m\mathcal{O}_D)^* = (\mathcal{O}_D/\Pi^{2m}\mathcal{O}_D)^*$ of this étale system. The space Σ is part of the first level of this étale system. The Galois group acting on this level is $(\mathcal{O}_D/\Pi\mathcal{O}_D)^* = (L^\circ/\pi L^\circ)^* = \ell^*$. Of course, this level is not an official member of the étale system as defined by Drinfel'd.

It follows that the Galois group of the system of Γ-invariant étale coverings of $\mathcal{Y}$ would be the group $\langle g \in \mathcal{O}_D^* \mid g \equiv 1 \mod \Pi\rangle$.

12.5 Speculation. *Generalisation.* We briefly sketch how the construction presented in this article could possibly be generalized to higher dimensions.

Let G be a semisimple algebraic group defined over K with building $\mathcal{B}_G$. Let us assume that there exists a semisimple algebraic group $H \subset G$ of the same rank $rk_K(G) = rk_K(H)$. Then the building $\mathbf{b}_H$ of the group $H(K)$ has the same rank as the building $\mathcal{B}_G$. A transversal covering $\mathcal{T}$ of the building $\mathcal{B}_G$ by $H(K)$-subbuildings consists of buildings $\mathbf{b} \cong \mathbf{b}_H$ such that non-empty intersections of buildings $\mathbf{b}, \mathbf{b}' \in \mathcal{T}$ are buildings of rank strictly less than $rk_K(G)$ and, moreover, each hyperspecial vertex $\mathbf{v} \in \mathcal{B}_G$ is contained in a $H(K)$-building $\mathbf{b} \in \mathcal{T}$.

Let us now assume that the group G is quasisplit, splits over an unramified extension of the field K and has a root system Φ_G with roots of different length. Let $\Phi_H \subset \Phi_G$ be the root system consisting of the longest roots in Φ_G. To the root system Φ_H belongs an algebraic group $H \subset G$ of rank $rk_K(H) = rk_K(G)$. In this situation it seems likely that Γ-invariant transversal coverings exist for discrete co-compact subgroups $\Gamma \subset G(K)$.

The long roots in Φ_G are either of type A or of type D. If the long roots are of type A, then the group $H(K)$ is a product of groups $SL(n, K)$. Therefore the group $H(K)$ acts on p-adic symmetric space Ω_H of dimension $rk_K(H)$. In this case it seems likely that one can construct a rigid analytic space $\mathcal{Y}$ for a discrete subgroup $\Gamma \subset G(K)$ such that $\mathcal{Y}/\Gamma$ is a proper algebraic variety.

To construct the analytic space $\mathcal{Y}$ one needs a suitable finite $H(K)$-equivariant étale covering $\Sigma_H \longrightarrow \Omega_H$. To allow glueing of the spaces Σ_H corresponding to buildings $\mathbf{b} \in \mathcal{T}$, the component of the reduction $\Sigma_{\mathbf{v},H}$ at a hyperspecial vertex $\mathbf{v} \in \mathbf{b} \subset \mathcal{B}_G$ of the space Σ_H must be a homogeneous variety for the group $G(\mathbb{F}_q)$.

The components of the reduction of the space Ω_H at hyperspecial vertices are compactifications of the Deligne-Lusztig variety $X(w_H)_H$ belonging to a Coxeter element w_H of $H(\mathbb{F}_q)$. The variety $X(w_H)_H$ for the group $SL(n, \mathbb{F}_q)$ equals $X(w_H)_H \cong \mathbb{P}^{n-1}_{\mathbb{F}_q} - \{\mathbb{F}_q - \textit{rational hyperplanes}\}$ (See [O-R] Introduction). It seems likely that the components of the reduction of the space Σ_H at hyperspecial vertices are compactifications of the Deligne-Lusztig variety $X(w_G)_G$ that belongs to a (twisted) Coxeter element w_G of the group $G(\mathbb{F}_q)$. A necessary (and probably sufficient) condition for the existence of such a variety Σ_H is the existence of a $H(\mathbb{F}_q)$-equivariant étale map $\varphi: \ X(w_G)_G \longrightarrow X(w_H)_H$ defined over some extension $\mathbb{F}_{q^d} \subset \mathbb{F}_q$ such that $\varphi(X(w_G)_G) \subset X(w_H)_H$ is an open subvariety. For the group $G(\mathbb{F}_q) = PU(3, \mathbb{F}_{q^2})$ this holds, since in this case $X(w_G)_G \cong \mathcal{H} - \mathcal{H}(\mathbb{F}_{q^2})$ and $X(w_H)_H = \mathbb{P}^1_{\mathbb{F}_q} - \mathbb{P}^1(\mathbb{F}_q)$.

Such a construction would give spaces $\mathcal{Y}$ for quasisplit semisimple algebraic groups that split over an unramified extension of K with root system $\Phi_G = B_3, BC_r, C_s, G_2,\ r \geq 1, s > 1$.

13 Examples

In §13.1 we consider the action of the groups $G_0 \cong S_3 \ltimes C^2_{q+1}$ and $G'_0 \cong C^2_{q+1}$ on the hermitian curve $\mathcal{H} \subset \mathbb{P}^2_\ell$. We describe their action on the isotropic points of $\mathbb{P}^2_\ell$ and determine the spreads invariant under the action of G_0 and G'_0.

In §13.2 we construct discrete co-compact subgroups $\Gamma, \Gamma' \subset PU(3, L)$ using the groups G_0 and G'_0. The groups Γ and Γ' act transitively on the hyperspecial vertices of the building $\mathcal{B}$. We determine the almost complete transversal coverings of $\mathcal{B}$ that are invariant under the action of the groups Γ and Γ'.

In §13.3 we study the quotients of spaces $\mathcal{Y}$ and $\mathcal{Y}'$ that correspond to the Γ- and Γ'-invariant transversal coverings of the building $\mathcal{B}$, respectively. Let $\Delta \subset \Gamma$ and $\Delta' \subset \Gamma'$ be normal subgroups of finite index and without elements of finite order such that $\Gamma/\Delta \cong G_0$ and $\Gamma'/\Delta' \cong G_0'$. We determine the genus of the curves $\mathcal{Y}/\Delta$ and $\mathcal{Y}'/\Delta'$. The quotients of $\mathcal{Y}/\Delta$ by G_0 and of $\mathcal{Y}'/\Delta'$ by G_0' are projective lines $\mathbb{P}^1_L$.

13.1 Finite groups and spreads

Let h_ℓ be the standard unitary form on ℓ^3 given by $h_\ell(x,y) = x_0y_0^q + x_1y_1^q + x_2y_2^q$. We study the action $G_0 \cong S_3 \ltimes C_{q+1}^2$ and $G_0' \cong C_{q+1}^2$ on the set of isotropic points in $\mathbb{P}^2_\ell$. We first consider the action of the group G_0 on the hermitian curve $\mathcal{H}$ defined by $h_\ell(x,x) = 0$. We also determine the spreads invariant under the action of G_0 and G_0'.

13.1 Proposition. *Let us assume that $p \neq 2$. Let $G_0 := S_3 \ltimes C_{q+1}^2$ act on the hermitian curve $\mathcal{H}$. Then $\mathcal{H}/G_0 = \mathbb{P}^1_\ell$. If $p \neq 3$ then the quotient map $\varphi : \mathcal{H} \longrightarrow \mathcal{H}/G_0 = \mathbb{P}^1_\ell$ has three branch points and branch groups $C_{2(q+1)}, C_3$ and C_2. If $p = 3$ the quotient map has two branch points with branch groups $C_{2(q+1)}$ and S_3.*

Proof. The quotient map decomposes as follows $\mathcal{H} \longrightarrow \mathcal{H}/C_{q+1}^2 \cong \mathbb{P}^1_\ell \longrightarrow \mathbb{P}^1_\ell/S_3 = \mathcal{H}/G_0 \cong \mathbb{P}^1_\ell$. The first part of the map gives three branch points $(0,1,-1), (1,0,-1), (1,-1,0) \in \mathbb{P}^1_\ell$ with branch groups C_{q+1}. Here we assume that the quotient $\mathbb{P}^1_\ell$ is given in a plane $\mathbb{P}^2_\ell$ by $x_0 + x_1 + x_2 = 0$.

If $p \neq 3$, then the quotient map $\mathbb{P}^1_\ell \longrightarrow \mathbb{P}^1_\ell/S_3$ has three branch points. The branch groups are C_2, C_2 and C_3. The ramification points with group C_2 are given by coordinate permutions of $(0,1,-1)$ and $(2,-1,-1)$. The ramification points of type $(0,1,-1)$ are branch points with group C_{q+1}. The group C_{q+1} acts only on the coordinate x_0 and the group C_2 fixes x_0 and permutes the coordinates x_1 and x_2. The stabilizer of an ℓ-valued isotropic point of $\mathcal{H}$ is contained in a Borel subgroup of the group $\mathrm{SU}(3,\ell)$. Since $p \nmid 2(q+1)$, the stabilizer is abelian. Further calculation shows that the group is cyclic of order $2(q+1)$. Therefore this gives a branch point with branch group $C_{2(q+1)}$. The points of type $(2,-1,-1)$ give a branch group C_2. The ramification group C_3 stabilizes the points $(1,\omega,\omega^2)$ and $(1,\omega^2,\omega)$ with $\omega^3 = 1$. Hence the group C_3 corresponds to a branch point.

If $p = 3$, then the group S_3 is a Borel group $B(2,1)$ of order 6. The quotient of $\mathbb{P}^1_\ell$ by the Borel group has two branch points. The branch groups are C_2 and S_3. The group S_3 stabilises the point $(1,1,1) \in \mathbb{P}^1_\ell$. The groups of type C_2 also fix points that are coordinate permutations of the point $(0,1,-1)$. The latter again results in a branch group $C_{2(q+1)}$ for the quotient of $\mathcal{H}$ by G. Therefore in this case we have two branch points and the branch groups are $C_{2(q+1)}$ and S_3. □

13.2 Table. In the table we describe the G_0-orbits on ℓ-valued isotropic points.

condition on q	stabilizer	♯ orbits	size	a point in the orbit
$p=3$	$C_{2(q+1)}$	1	$3(q+1)$	$(0,1,a),\ a^{q+1}=-1$
	S_3	1	$(q+1)^2$	$(1,1,1)$
	1	$(q-3)/6$	$6(q+1)^2$	
$q \equiv 1 \mod 3$	$C_{2(q+1)}$	1	$3(q+1)$	$(0,1,a),\ a^{q+1}=-1$
	C_2	1	$3(q+1)^2$	$(1,1,b),\ b^{q+1}=-2$
	C_3	1	$2(q+1)^2$	$(1,c,c^2),\ c^{q+1}=\omega,\ \omega^3=1$
	1	$(q-7)/6$	$6(q+1)^2$	
$q \equiv 2 \mod 3$	$C_{2(q+1)}$	1	$3(q+1)$	$(0,1,a),\ a^{q+1}=-1$
	C_2	1	$3(q+1)^2$	$(1,1,b),\ b^{q+1}=-2$
	1	$(q-5)/6$	$6(q+1)^2$	

13.3 Proposition. *i) If $p=3$, then a maximal G_0-invariant partial spread consists of q^2-2q lines. There are $3^{\frac{q-3}{6}}$ such maximal partial spreads.*

ii) If $q \equiv 1 \mod 3$, then a maximal G_0-invariant partial spread consists of q^2-3q-1 lines. There are $3^{\frac{q-7}{6}}$ such partial spreads.

iii) If $q \equiv 2 \mod 3$, then there exist $3^{\frac{q-5}{6}}$ distinct G_0-invariant spreads.

Proof. We will first show that a G_0-invariant (partial) spread can contain only lines $a^\perp$ such that some coordinate $a_i = 0$ for $i=0,1,2$. Then we will determine for each G_0-orbit of isotropic points which choices of lines $a^\perp$ are possible and whether they are contained in the partial spread or not.

Let $a \in \mathbb{P}^2(\ell)$ be an anisotropic point. Let us assume that the line $a^\perp$ is contained in a G_0-invariant (partial) spread. Then some coordinate a_i, $i=1,2,3$ is zero. Indeed, if all coordinates $a_i \neq 0$, then the orbit $G_0 \cdot a$ consists of at least $(q+1)^2 > q^2-q+1$ points. Since a spread consists of q^2-q+1 lines, this cannot be.

Let $a \in \mathbb{P}^2(\ell)$ be an anisotropic point with at least one coordinate equal to zero. The isotropic points in the line $a^\perp$ are contained in a single G_0-orbit $G_0 \cdot y$. It follows that the isotropic points in the lines in the orbit $G_0 \cdot a^\perp$ form the orbit $G_0 \cdot y$.

Let $a' \in \mathbb{P}^2(\ell)$ be another anisotropic point with at least one zero coordinate. Let us assume that the intersection $a^\perp \cap a'^\perp$ is an isotropic point. Then the isotropic points in orbits $G_0 \cdot a^\perp$ and $G_0 \cdot a'^\perp$ cover the same G_0-orbit $G_0 \cdot y$. In particular, choosing for each G_0-orbit $G_0 \cdot y$ of isotropic points an anisotropic point $a \perp y$ is almost equivalent to defining a (partial) spread. The only condition required is that none of the lines in the orbits $G_0 \cdot a^\perp$ intersect in an isotropic point.

Let us now determine the size of the orbits $G_0 \cdot a$ for anisotropic points $a \in \mathbb{P}^2(\ell)$ such that some coordinate $a_i = 0$ for $i = 0, 1, 2$. If $a \in G_0 \cdot (1, 0, 0)$, then the orbit has size 3 and if $a \in G_0 \cdot (1, -1, 0)$, then the orbit consists of $3(q + 1)$ points. In all other cases the orbit $G_0 \cdot a$ contains $6(q + 1)$ points.

All we have to do now is to see in which cases for $y \in a^\perp$ the equality $(q + 1) \cdot |G_0 \cdot a| = |G_0 \cdot y|$ holds. For this we use table 13.2. If the orbit $G_0 \cdot y$ consists of points stabilised by a group $C_{2(q+1)}$, then $y_i = 0$ for some $i \in \{0, 1, 2\}$. In particular, $|G_0 \cdot y| = 3(q + 1)$ and we need to take $a \in G_0 \cdot (1, 0, 0)$. If the orbit $G_0 \cdot y$ consists of points stabilised by a group C_2, then $|G_0 \cdot y| = 3(q + 1)^2$ and we need to take $a \in G_0 \cdot (1, -1, 0)$. If the orbit $G_0 \cdot y$ consists of points stabilised by a group C_3 if $q \equiv 1 \mod 3$ or of points stabilised by a group S_3 if $p = 3$, then no choice of a orbit $G_0 \cdot a$ has the correct size. Hence these points cannot be contained in a maximal partial spread.

If the orbit $G_0 \cdot y$ consists of points with trivial stabiliser, then it has size $6(q + 1)^2$ and $a \notin G_0 \cdot (1, 0, 0), G_0 \cdot (1, -1, 0)$ gives the same size. In this case all the coordinates $y_i \neq 0$ and we have three possibilities for the choice of the anisotropic point $a \perp y$.

Therefore the number of maximal G_0-invariant (partial) spreads equals 3^m, where m is the number of G_0-orbits of isotropic points with trivial stabiliser. Moreover, the isotropic points that are covered by the (partial) spread have stabiliser $C_{2(q+1)}, C_2$ or 1. In particular, only for $q \equiv 2 \mod 3$ do full spreads exist. The proposition follows from this and table 13.2. □

13.4 Proposition. *Let G_0' be the group $G_0' \cong C_{q+1}^2$. Then there exist 3^{q-2} G_0'-invariant spreads.*

Proof. Let $a \in \mathbb{P}^2(\ell)$ be an anisotropic point such that the line $a^\perp$ is contained in a G_0'-invariant spread. Then some coordinate $a_i = 0$ with $i \in \{0,1,2\}$. If none of the coordinates a_i are zero, then $|G_0' \cdot a| = (q+1)^2 > q^2 - q + 1$. Hence this cannot be. Furthermore all the isotropic points in the line $a^\perp$ are contained in a single G_0'-orbit.

Therefore we can use the same methods as in the proof of prop. 13.3 above. The size of orbits $G_0' \cdot a$ is 1 if $a \in \{(1,0,0),(0,1,0),(0,0,1)\}$ and $q+1$ in all other cases. The size of orbits $G_0' \cdot y$ of isotropic points is $q+1$ if $y_i = 0$ for some $i \in \{0,1,2\}$ and $(q+1)^2$ in all other case. Now $(q+1) \cdot |G_0' \cdot a| = |G_0' \cdot y|$ with $y \in a^\perp$ holds for all G_0'-orbits. Again there are three choices possible for $a \perp y$ if all the coordinates $y_i \neq 0$. There are $\frac{q^3+1-3(q+1)}{(q+1)^2} = q-2$ distinct orbits of isotropic points with non-zero coordinates. Therefore there exist 3^{q-2} distinct G_0'-invariant spreads.

TO DO: Isomorphism under S_3-action??? $\square$

13.5 Remark. If $q \equiv 2 \mod 3$, then the set of lines $\{g((1,0,0)^\perp), g((1,1,0)^\perp) \mid g \in G_0\}$ is a minimal G_0-adapted partial spread. If $p = 3$ or $q \equiv 1 \mod 3$, then no G_0-adapted partial spread exists.

The set of lines $\{(1,0,0)^\perp, (0,1,0)^\perp, (0,0,1)^\perp\}$ is a minimal G_0'-adapted partial spread.

13.2 Discrete groups and transversal coverings

We use the groups G_0 and G_0' to construct amalgams Γ and Γ'. We embed these amalgams into the group $PU(3,L)$ such that the groups act transitively on the hyperspecial vertices of the building $\mathcal{B}$. We determine the complete transversal coverings invariant under the action of the groups Γ and Γ'.

13.6 Definition. Let V be the vector space $V := L^3$ with on it the standard unitary form $h(x,y)$ given by $h(x,y) = x_0\overline{y_0} + x_1\overline{y_1} + x_2\overline{y_2}$. Let $v \in V$ be a vector. We define a map $\rho_v : L^3 \longrightarrow L^3$ by $\rho_v(x) = x + (\zeta_{q+1} - 1) \cdot \frac{h(x,v)}{h(v,v)} \cdot v$. The element ρ_v multiplies the vector v by ζ_{q+1} and acts trivially on $v^\perp$. The element ρ_v preserves the unitary form $h(x,y)$ and is an element of the group $PU(3,L)$ acting on $\mathbb{P}(V)$.

We will use the elements ρ_v for suitable vectors $v \in V$ together with the group $G_0 \cong S_3 \ltimes C_{q+1}^2$ to define a discrete group $\Gamma \subset PU(3,L)$. To avoid edges $\mathbf{e} \ni \mathbf{v}_0$ stabilised by subgroups C_3 we assume that $q \equiv -1 \mod 3$.

The $\Gamma_{\mathbf{v}_0}$-orbits of the anisotropic points $(1,0,0)$ and $(1,1,0)$ consist of 3 and $3(q+1)$ vectors, respectively. These vectors have stabiliser $C_2 \ltimes C_{q+1}^2$ and

$C_2 \times C_{q+1}$, respectively. To obtain a group Γ that acts transitively on vertices $\mathbf{v}$ of type $\tau(\mathbf{v}) = 0$, we add to the stabilizer of each vertex $\mathbf{v}$ neighbouring $\mathbf{v}_0$ a group C_{q+1}, that acts transitively on the edges $\mathbf{e} \ni \mathbf{v}$.

Let Γ be the amalgam of $G_0 \cong S_3 \ltimes C_{q+1}^2$ with a group $C_2 \ltimes C_{q+1}^2$ along $C_{2(q+1)}$, with a group $C_2 \times C_{q+1}$ along C_2 and with $\frac{q-5}{6}$ groups C_{q+1} along 1. If $q \equiv 2 \mod 3$, then the amalgam Γ is defined as being:

$$\Gamma := (C_2 \ltimes C_{q+1}^2) *_{C_{2(q+1)}} (S_3 \ltimes C_{q+1}^2) *_{C_2} (C_2 \times C_{q+1}) \underbrace{* C_{q+1} \ldots\ldots * C_{q+1}}_{\frac{q-5}{6}}$$

If $p = 3$, then we define a similar amalgam Γ as follows:

$$\Gamma := (C_2 \ltimes C_{q+1}^2) *_{C_{2(q+1)}} (S_3 \ltimes C_{q+1}^2) *_{S_3} (S_3 \times C_{q+1}) \underbrace{* C_{q+1} \ldots\ldots * C_{q+1}}_{\frac{q-3}{6}}$$

13.7 Proposition. *i) If $q \equiv 2 \mod 3$, then the group Γ can be embedded into $PU(3, L)$ as a discrete co-compact subgroup that acts transitively on the hyperspecial vertices of $\mathcal{B}$.*

ii) If $p = 3$ and the extension $K \supset \mathbb{Q}_3$ is unramified, then the group Γ can be embedded into $PU(3, L)$ as a discrete co-compact subgroup that acts transitively on the hyperspecial vertices of $\mathcal{B}$.

Proof. Let us first prove statement (i). Let the equivalence class $[M_0] := [\langle e_0, e_1, e_2 \rangle]$ correspond to the vertex $\mathbf{v}_0 \in \mathcal{B}$. Let $\Gamma_{\mathbf{v}_0}$ be the group $\Gamma_{\mathbf{v}_0} \cong S_3 \ltimes C_{q+1}^2$. Then $\Gamma_{\mathbf{v}_0}$ acts on M_0 and preserves the unitary form $h(x, y)$.

The edges $\mathbf{e} \ni \mathbf{v}_0$ correspond to the isotropic points in $\mathbb{P}(M_0 \otimes \ell)$. The module M_1 that corresponds to a neighbouring vertex $\mathbf{v}_1 \in \mathcal{B}$ is given by $M_1 := \langle \frac{v}{\pi}, M_0 \rangle$. Here the vector $v \in M_0$ is choosen such that the reduction $v \mod \pi$ is the isotropic point that corresponds to the edge $\mathbf{e} \ni \mathbf{v}_0, \mathbf{v}_1$ and $h(v, v) = \pi$ holds. The element $\rho_v \in PU(3, L)$ preserves the module M_1 and acts transitively on the edges $\mathbf{e} \in \mathbf{v}_1$.

The element ρ_v depends on the choice of the vector $v \in M_0$. To embed the amalgam Γ into $PU(3, L)$ as a discrete subgroup we need to impose some conditions on the choice of the vectors v. The choice of the $q^3 + 1$

vectors v has to be G_0-invariant. For a vertex $\mathbf{v}_1$ in a G_0-orbit such that the edge $\mathbf{e} \ni \mathbf{v}_0, \mathbf{v}_1$ has trivial stabilizer, this causes no problems. One chooses a suitable vector v for a vertex $\mathbf{v}_1$ in the orbit and for the vertices $g(\mathbf{v}_1)$, $g \in G_0$, one uses the vector $g(v)$.

For a vertex $\mathbf{v}_1$ such that the stabilizer $G_{\mathbf{e}} \subset G_0$ of the edge $\mathbf{e} \ni \mathbf{v}_0, \mathbf{v}$ is non-trivial, one must choose the vector v such that the group $\langle \rho_v, G_{\mathbf{e}} \rangle$ is finite and as defined in the amalgam Γ.

Let us first consider the edge $\mathbf{e}$ with $G_{\mathbf{e}} = C_{2(q+1)}$. Let $\alpha \in (L^\circ)^*$ be a unit root such that $\alpha^{q+1} = -1$. We consider the edge $\mathbf{e} \ni \mathbf{v}_0$ that corresponds to the isotropic vector $(0, 1, \alpha) \mod \pi$. Then $G_{\mathbf{e}} = \langle g_\alpha \rangle$, where g_α acts by $g_\alpha((x_0, x_1, x_2)) = (x_0, \alpha^{q-1} x_2, -x_1)$.

The vector $v \in M_0 \cong (L^\circ)^3$ can be choosen to be $v = (0, 1, v_2)$ orthogonal to the vector $(1, 0, 0)$. We take v_2 to be $v_2 = \alpha(1 + \pi u)$, with $u \in (K^\circ)^*$. Then $g_\alpha(v) = (0, \alpha^q(1 + \pi u), -1) = (0, \overline{v_2}, -1)$. Therefore $h(v, g_\alpha(v) = 0$ and the elements ρ_v, $\rho_{g_\alpha(v)}$ and g_α^2 commute. In particular, $\langle \rho_v, G_{\mathbf{e}} \rangle \cong C_2 \ltimes C_{q+1}^2$ holds.

Let us now consider the case $G_{\mathbf{e}} = C_2$. The vector v can be choosen to be orthogonal to the vector $(1, -1, 0)$. Then $v = (1, 1, v_2)$ such that $v_2^{q+1} \equiv -2 \mod \pi$. The vector v is stabilized by the group C_2 that permutes the coordinates x_0 and x_1. In particular, $\langle \rho_v, G_{\mathbf{e}} \rangle \cong C_2 \times C_{q+1}$.

For $q \equiv -1 \mod 3$ the groups $\Gamma_{\mathbf{v}_1} := \langle \rho_v, G_{\mathbf{e}} \rangle$, $d_{\mathcal{B}}(\mathbf{v}_1, \mathbf{v}_0) = 1$, together with $\Gamma_{\mathbf{v}_0} \cong G_0$ define an embedding of the amalgam Γ into $PU(3, L)$ as a discrete subgroup. For each vertex $\mathbf{v}_1 \in \mathcal{B}$ that forms an edge with the vertex $\mathbf{v}_0$, the added groups $\rho_v \cong C_{q+1}$ act transitively on the edges $\mathbf{e} \ni \mathbf{v}_1$. In particular, the embedding of the group Γ is co-compact and acts transitively on the hyperspecial vertices in $\mathcal{B}$.

The proof of statement (ii) is similar to that of statement (i). The only difference is the choice of the vector v that is stabilised by a group S_3. Let the group S_3 act on the coordinates by permutation. Then the only vector v fixed by the group S_3 is the vector $v = (1, 1, 1)$. Then $h(v, v) = 3$. Therefore it is necessary that $\pi = 3$ holds. In particular, for $p = 3$ the embedding of the group Γ only acts transitively on the hyperspecial vertices, if the extension $K \supset \mathbb{Q}_3$ is unramified. $\square$

13.8 Remark. *Arithmeticity.* For $q = 3, 5$ the groups Γ are isomorphic to an arithmetic group. Let $q = 3$ and let $L := \mathbb{Q}_3(i)$ with $i^2 = -1$. The group Γ equals $\Gamma = (C_2 \ltimes (C_4)^2) *_{C_8} (S_3 \ltimes (C_4)^2) *_{S_3} (S_3 \times C_4)$. Let Λ be the lattice $\Lambda := \oplus_{j=0}^2 \mathbb{Z}[i] \cdot e_j$. To embed the group Γ, one can use the vectors $(1+i, 1, 0)$

and $(1,1,1)$ to define the elements ρ_v. Then $\Gamma \cong PU(\Lambda[\frac{1}{3}])$.

Let $q = 5$ and let ω be a third root of unity, i.e. $\omega^3 = 1$. Let $L = \mathbb{Q}_5(\omega) = \mathbb{Q}_5(\sqrt{-3})$. One has $\Gamma = (C_2 \ltimes (C_6)^2) *_{C_{12}} (S_3 \ltimes (C_6)^2) *_{C_2} (C_2 \times C_6)$. Let Λ be the lattice $\Lambda := \oplus_{j=0}^{2} \mathbb{Z}[\omega] \cdot e_j$. To embed the group Γ, one can use the vectors $(2,1,0)$ and $(1-\omega,1,1) \in \Lambda$ to define the elements $\rho_v \in \Gamma$. Then $\Gamma \cong \mathrm{PU}(\Lambda[\frac{1}{5}])$.

13.9 Definition. *The group* Γ'. Let the group Γ' be the amalgam of $G_0' \cong C_{q+1}^2$ with three groups C_{q+1}^2 along groups C_{q+1} and with $q-2$ groups C_{q+1} along 1. The amalgam Γ' is defined as being:

$$\Gamma' := C_{q+1}^2 *_{C_{q+1}} \begin{array}{c} C_{q+1}^2 \\ *_{C_{q+1}} \\ C_{q+1}^2 \\ * \ldots * \\ \underbrace{C_{q+1} \quad C_{q+1}}_{q-2} \end{array} *_{C_{q+1}} C_{q+1}^2$$

13.10 Remark. *Relation between the groups* Γ *and* Γ'. If $q \not\equiv 1 \mod 3$, then there exists a map $\Gamma \longrightarrow S_3$, such that the kernel is isomorphic to Γ'. Using remark 13.8, it follows that the group Γ' is isomorphic to an arithmetic group if $q = 3$ and $q = 5$.

13.11 Proposition. *The group* Γ' *can be embedded into* $PU(3, L)$ *as a discrete co-compact subgroup that acts transitively on the hyperspecial vertices of* $\mathcal{B}$.

Proof. Similar to the proof of prop. 13.7. □

13.12 Proposition. *i) If* $q \equiv 2 \mod 3$, *then there is a bijection between* G_0*-invariant spreads and* Γ*-invariant complete transversal coverings.*

ii) There exists a bijection between G_0'*-invariant spreads and* Γ'*-invariant complete transversal coverings.*

Proof. We only prove the proposition for Γ and G_0. The proof for Γ' and G_0' is entirely identical.

So let us assume that $q \equiv 2 \mod 3$. We first define a map from Γ-invariant transversal coverings to G_0-invariant spreads and a map from G_0-invariant spreads to Γ-invariant transversal coverings. Then we will show that the maps indeed define a bijection.

We fix a G_0-equivariant bijection between edges $\mathbf{e} \ni \mathbf{v}_0$ and isotropic points in $\mathbb{P}^2(\ell)$. The edges $\mathbf{e} \ni \mathbf{v}_0$ contained in a subbuilding $\mathbf{b} \subset \mathcal{B}$ are contained a line $a_{\mathbf{b}}^{\perp} \cong \mathbb{P}^1_{\ell}$. The buildings $\mathbf{b} \ni \mathbf{v}_0$ contained in a Γ-invariant complete transversal covering $\mathcal{T}$ define a set of lines $\{a_{\mathbf{b}}^{\perp} \mid \mathbf{b} \in \mathcal{T},\ \mathbf{v}_0 \in \mathbf{b}\}$. Since $\mathcal{T}$ is a Γ-invariant complete transversal covering, the set of lines defines a G_0-invariant spread.

Let $\mathcal{S}$ be a G_0-invariant spread. Then each isotropic point $y \in \mathbb{P}^2(\ell)$ is contained in a unique line $a_y^{\perp} \in \mathcal{S}$. The point $a_y \in \mathbb{P}^2(\ell)$ is anisotropic with at least one coordinate equal to zero. Let $\mathbf{e} \ni \mathbf{v}_0$ be the edge that corresponds to the isotropic point y. Then define the building $\mathbf{b} \in \mathcal{T}$ containing the edge $\mathbf{e} \ni \mathbf{v}_0$ as being the building of the stabiliser of the anisotropic $x_{\mathbf{b}}$. The point $x_{\mathbf{b}}$ is the unique point such that the coordinates $(x_{\mathbf{b}})_i = 0$ and, moreover $h(x_{\mathbf{b}}, v) = 0$. Here v is the vector that defines the element $\rho_v \in \Gamma$ that preserves the vertex $\mathbf{v}_1 \in \mathbf{e}$.

To prove that the maps indeed define a bijection it is sufficient to show that the only Γ-invariant transversal coverings are the ones given by the map. Let us assume that $\mathcal{T}$ is not as given by the second map. Then there either exists a building $\mathbf{b} \in \mathcal{T}$, $\mathbf{v}_0 \in \mathbf{b}$ such that $x_{\mathbf{b}}$ has only non-zero coordinates or $h(x_{\mathbf{b}}, v) \equiv 0 \mod \pi$ but not $h(x_{\mathbf{b}}, v) = 0$. In the first case $\Gamma_{\mathbf{v}_0} \cdot \mathbf{b}$ consists of more than $q^2 - q + 1$ buildings all containing the vertex $\mathbf{v}_0$. Hence this cannot be.

In the second case $\Gamma_{\mathbf{v}_1} \cdot \mathbf{b}$ consists of more than one building and these buildings contain all the edges $\mathbf{e} \ni \mathbf{v}_1$. This cannot be. Hence the two maps defined above together give indeed a bijection. □

13.13 Proposition. *Let Γ and Γ' be embedded into $PU(3, L)$ as in prop. 13.7 and 13.11.*

i) If $q \equiv 2 \mod 3$, then the quotient $\mathcal{B}/\Gamma$ is a tree that consists of one hyperspecial vertex $\mathbf{v}_0$ and $\frac{q+7}{6}$ non-hyperspecial vertices that form an edge with the vertex $\mathbf{v}_0$.

ii) The quotient $\mathcal{B}/\Gamma'$ is a tree that consists of one hyperspecial vertex $\mathbf{v}_0$ and $q+1$ non-hyperspecial vertices that form an edge with the vertex $\mathbf{v}_0$.

Proof. This follows directly from the fact that the groups Γ (with $q \equiv 2 \mod 3$) and Γ' act transitively on hyperspecial vertices in $\mathcal{B}$ and from the action of G_0 and G_0' on isotropic points. □

13.14 Definition. *Normal subgroups without elements of finite order.* Let $\varphi: \ \Gamma \to \Gamma_{\mathbf{v}_0} \cong G_0$ be a map such that no element of finite order is in the kernel. Since all the elements $\rho_v \in \Gamma$ have order $q+1$, such a map φ exists. Let $\Delta \subset \Gamma$ be the kernel $\Delta := ker(\varphi)$. By construction the group Δ contains no elements of finite order. Moreover, the group Δ acts transitively on hyperspecial vertices $\mathbf{v} \in \mathcal{B}$.

Let $\varphi': \ \Gamma' \to \Gamma'_{\mathbf{v}_0} \cong G'_0$ be a map such that no element of finite order is in the kernel. Since all the elements $\rho_v \in \Gamma'$ have order $q+1$, such a map φ exists. Let $\Delta' \subset \Gamma'$ be the kernel $\Delta := ker(\varphi)$. The group Δ' contains no elements of finite order and acts transitively on hyperspecial vertices $\mathbf{v} \in \mathcal{B}$.

13.15 Proposition. *i) If $q \equiv 2 \mod 3$, then the quotient $\mathcal{B}/\Delta$ consists of one hyperspecial vertex $\mathbf{v}_0$ and q^2-q+1 non-hyperspecial vertices that form $q+1$ edges with the vertex $\mathbf{v}_0$.*

ii) The quotient $\mathcal{B}/\Delta'$ consists of one hyperspecial vertex $\mathbf{v}_0$ and q^2-q+1 non-hyperspecial vertices that form $q+1$ edges with the vertex $\mathbf{v}_0$.

Proof. Direct calculation. □

13.16 Definition. *Minimal Γ- and Γ'-adapted transversal coverings.* One can use the minimal G_0-adapted and G'_0-adapted partial spreads to define Γ-adapted and Γ'-adapted transversal which are non-complete. Let $q \equiv 2 \mod 3$. Then we define $\mathcal{T}_{min}$ as being the transversal covering of $\mathcal{B}$ defined by the vectors $\{\gamma((1,0,0)), \gamma((1,1,0)) \mid \gamma \in \Gamma\}$. Then $\mathcal{T}_{min}$ is a minimal Γ-adapted transversal covering.

We define $\mathcal{T}'_{min}$ as being the transversal covering of $\mathcal{B}$ defined by the vectors $\{\gamma((1,0,0)), \gamma((0,1,0)), \gamma((0,0,1)) \mid \gamma \in \Gamma'\}$. Then $\mathcal{T}'_{min}$ is a minimal Γ'-adapted transversal covering.

13.3 Algebraic curves

We use the groups $\Delta \subset \Gamma$ and $\Delta' \subset \Gamma'$ to construct algebraic curves. We assume that the groups Γ and Γ' are embedded into $PU(3, L)$ as described in prop. 13.7 and 13.11. Both the complete transversal covering and the minimal Γ- and Γ'-adapted transversal coverings are used. We determine in each case the genus and show that the quotient of the curve by the group Γ/Δ (resp. Γ'/Δ') is a projective line $\mathbb{P}^1_L$.

13.17 Proposition. *Let $q \equiv 2 \mod 3$ and let $\mathcal{T}$ be a complete Γ-invariant transversal covering and let $\mathcal{Y}$ be the corresponding analytic space.*

i) $\mathcal{Y}/\Gamma \cong \mathbb{P}^1_L$.

ii) The reduction of the quotient $\mathcal{Y}/\Delta$ consists of q^2-q+2 components $\mathcal{H}$. Each component corresponding to a non-hyperspecial vertex intersects the unique component $\mathcal{H}$ corresponding to a hyperspecial vertex in $q+1$ distinct ℓ-valued points.

iii) The quotient map $\mathcal{Y}/\Delta \longrightarrow \mathcal{Y}/\Gamma \cong \mathbb{P}^1_L$ has $\frac{q^2-q+22}{6}$ branch points. There are two branch points with group C_2, one point with group C_3 and $\frac{q^2-q+4}{6}$ points with group C_{q+1}.

iv) $g(\mathcal{Y}/\Delta) = \frac{q^4+q^2}{2}$.

Proof. The quotient of the hermitian curve $\mathcal{H}$ by a group C_{q+1} or C_{q+1}^2 where the groups C_{q+1} stabilize an anisotropic vector is a $\mathbb{P}^1_\ell$. Since the stabilizer of each component $\mathcal{H}$ contains a normal subgroup of this type, the quotient of each component is a curve $\mathbb{P}^1_\ell$. Since the quotient graph $\mathcal{B}/\Gamma$ is a tree, statement (i) follows.

Statement (ii) is clear.

Let us now consider statement (iii). The branch groups of the map $\mathcal{Y}/\Delta \longrightarrow \mathcal{Y}/\Gamma$ are finite and stabilise a vertex $\mathbf{v} \in \mathcal{B}$. Therefore they equal branch groups of the maps $\mathcal{H} \longrightarrow \mathcal{H}/\Gamma_{\mathbf{v}} \cong \mathbb{P}^1_\ell$. A branch point of the map $\mathcal{H} \longrightarrow \mathcal{H}/\Gamma_{\mathbf{v}}$ is contained in the generic fibre of $\mathcal{Y}$ (or $\mathcal{Y}/\Delta$) if and only if the corresponding branch group does not stabilise an edge $\mathbf{e} \ni \mathbf{v}$. To prove (iii) it is therefore sufficient to determine the branch points and groups for each component of the reduction of $\mathcal{Y}/\Gamma$.

The quotient map $\mathcal{H} \longrightarrow \mathcal{H}/G_0 = \mathbb{P}^1_\ell$ has three branch points and branch groups $C_{2(q+1)}, C_3$ and C_2. The groups $C_{2(q+1)}$ and C_2 stabilise an edge $\mathbf{e}$ and the corresponding points are omitted from the generic fibre of $\mathcal{Y}$. Therefore the corresponding component $\mathbb{P}^1_\ell$ of the reduction of $\mathcal{Y}/\Gamma$ gives a single branch point stabilized by a group C_3.

The quotient map $\mathcal{H} \longrightarrow \mathcal{H}/(C_2 \ltimes C_{q+1}^2) = \mathbb{P}^1_\ell$ has three branch points and branch groups $C_{2(q+1)}, C_{q+1}$ and C_2. The group $C_{2(q+1)}$ stabilises an edge $\mathbf{e}$ and the corresponding point is omitted from the generic fibre of $\mathcal{Y}$. Hence the corresponding component $\mathbb{P}^1_\ell$ of the reduction of $\mathcal{Y}/\Gamma$ contributes one branch point with group C_2 and one branch point with group C_{q+1}.

The component $\mathbb{P}^1_\ell$ in $\mathcal{Y}/\Gamma$ coming from a map $\mathcal{H} \longrightarrow \mathcal{H}/(C_2 \times C_{q+1})$ gives one branch point with group C_2 and $\frac{q+1}{2}$ points with branch group C_{q+1}. The $\frac{q-5}{6}$ components $\mathbb{P}^1_\ell$ in $\mathcal{Y}/\Gamma$ coming from a map $\mathcal{H} \longrightarrow \mathcal{H}/C_{q+1}$ give each $q+1$ branch points with group C_{q+1}. We have now determined the contribution of each component of the reduction of $\mathcal{Y}/\Gamma$ to the set of branch points and groups of the map $\mathcal{Y}/\Delta \longrightarrow \mathcal{Y}/\Gamma$. Statement (ii) is obtained by addition of these contributions.

Statement (iv) follows from statement (iii) and the Riemann-Hurwitz formula or from statement (ii) and the formula for the genus obtained in prop. 11.8. □

13.18 Proposition. *Let* $q \equiv 2 \mod 3$ *and let* $\mathcal{Y}_{min}$ *be the rigid space corresponding to the minimal* Γ*-adapted transversal covering* $\mathcal{T}_{min}$*.*

- *i) The quotient* $\mathcal{Y}_{min}/\Gamma$ *is a projective line* $\mathbb{P}^1_L$*.*
- *ii) The reduction of the quotient* $\mathcal{Y}_{min}/\Delta$ *consists of* $3q+7$ *components* $\mathcal{H}$ *and* q^2-4q-5 *components* $\mathbb{P}_\ell$*. Each component corresponding to a non-hyperspecial vertex intersects the unique component* $\mathcal{H}$ *corresponding to a hyperspecial vertex in* $q+1$ *distinct* ℓ*-valued points.*
- *iii) The quotient map* $\mathcal{Y}_{min}/\Delta \longrightarrow \mathcal{Y}_{min}/\Gamma \cong \mathbb{P}^1_L$ *has* $\frac{5q+17}{6}$ *branch points. There are* $\frac{5q-1}{6}$ *branch points with branch group* C_{q+1}*, one point with branch group* C_3 *and two points with branch group* C_2*.*
- *iv) The genus of the quotient* $\mathcal{Y}_{min}/\Delta$ *is* $g(\mathcal{Y}_{min}/\Delta) = \frac{5q^3+2q^2-5q}{2}$*.*

Proof. The quotient of the hermitian curve $\mathcal{H}$ by a group C_{q+1} or C^2_{q+1} where the groups C_{q+1} stabilize an anisotropic vector is a $\mathbb{P}^1_\ell$. Since the stabilizer of each component $\mathcal{H}$ contains a normal subgroup of this type, the quotient of each component is a curve $\mathbb{P}^1_\ell$. Since the quotient graph $\mathcal{B}/\Gamma$ is a tree, statement (i) follows.

To prove statement (ii), one observes that the vertex $\mathbf{v}_0$ is contained in $3(q+2)$ buildings $\mathbf{b} \in \mathcal{T}$. Therefore quotient $\mathcal{Y}_{min}/\Delta$ consists of $1+3(q+2) = 3q+7$ components $\mathcal{H}$ and $\frac{q^3+1}{q+1} - 3(q+2) = q^2-4q-5$ components $\mathbb{P}_\ell$.

Let us now prove statement (iii) by determining the contribution of each component of the reduction of $\mathcal{Y}_{min}/\Gamma$ The component stabilized by the group $S_3 \ltimes C^2_{q+1}$ gives a single branch point stabilized by a group C_3. The component stabilized by a group $C_2 \ltimes C^2_{q+1}$ gives one branch point with group C_2 and one branch point with group C_{q+1}. The component stabilized by a group $C_2 \times C_{q+1}$ gives one branch point with group C_2 and $\frac{q+1}{2}$ points with branch group C_{q+1}. The $\frac{q-5}{6}$ components $\mathbb{P}^1_\ell$ stabilized by a group C_{q+1} give each two branch points with group C_{q+1}.

Statement (iv) follows from statement (iii) and the Riemann-Hurwitz formula or from statement (ii) and the formula for the genus obtained in prop. 11.8. $\square$

13.19 Proposition. *Let $\mathcal{T}'$ be a complete Γ'-invariant transversal covering and let $\mathcal{Y}'$ be the corresponding analytic space.*

i) $\mathcal{Y}'/\Gamma' \cong \mathbb{P}^1_L$.

ii) The reduction of the quotient $\mathcal{Y}'/\Delta'$ consists of q^2-q+2 components $\mathcal{H}$. Each component corresponding to a non-hyperspecial vertex intersects the unique component $\mathcal{H}$ corresponding to a hyperspecial vertex in $q+1$ distinct ℓ-valued points.

iii) The quotient map $\mathcal{Y}'/\Delta' \longrightarrow \mathcal{Y}'/\Gamma' \cong \mathbb{P}^1_L$ has q^2-q+4 branch points with branch group C_{q+1}.

iv) $g(\mathcal{Y}'/\Delta') = \frac{q^4+q^2}{2}$.

Proof. The proof of statements (i) and (ii) is as in the proposition above. Let us now prove statement (iii) by determining the contribution of each component of the reduction of $\mathcal{Y}'/\Gamma'$. The group $\Gamma'_{\mathbf{v}_0}$ acts without fixed points on the component $\mathcal{H}$. The three non-hyperspecial vertices $\mathbf{v}_1$ with stabiliser C^2_{q+1} give each two branch point with stabiliser C_{q+1}. The $q-2$ components $\mathcal{H}$ corresponding to a non-hyperspecial vertex $\mathbf{v}_1$ with stabiliser $\Gamma_{\mathbf{v}_1} \cong C_{q+1}$ give each $q+1$ branch points with stabiliser C_{q+1}. Hence we have in total q^2-q+4 branch points with branch group C_{q+1}.

To prove statement (iv) we use the formula $g(\mathcal{Y}'/\Delta') = 1 + n_{\mathbf{e}} + \sum_{\mathbf{v}}(g_{\mathbf{v}} - 1) = \frac{q^4+q^2}{2}$ $\square$

13.20 Proposition. *Let $\mathcal{Y}'_{min}$ the rigid space belonging to the minimal Γ'-adapted transversal covering $\mathcal{T}'_{min}$. The following statements hold.*

i) $\mathcal{Y}'_{min}/\Gamma' \cong \mathbb{P}^1_L$.

ii) *The reduction of the quotient* $\mathcal{Y}'_{min}/\Delta'$ *consists of* 4 *hermitian curves* $\mathcal{H}$ *and* q^2-q-2 *lines* $\mathbb{P}^1_\ell$. *Each component intersects the hermitian curve* $\mathcal{H}$ *that corresponds to the vertex* $\mathbf{v}_0$ *in* $q+1$ *distinct* ℓ *valued (isotropic) points.*

iii) *The quotient map* $\mathcal{Y}'_{min}/\Delta' \longrightarrow \mathcal{Y}'_{min}/\Gamma' \cong \mathbb{P}^1_L$ *has* $2(q+1)$ *branch points with branch group* C_{q+1}.

iv) $g(\mathcal{Y}'_{min}/\Delta') = q^3 + q^2 - q$.

Proof. Statement (i) follows from the fact that the quotient of a hermitian curve $\mathcal{H}$ by a group C_{q+1} is projective line $\mathbb{P}^1_L$. Indeed, it follows that the reduction of quotient $\mathcal{Y}'_{min}/\Gamma$ is a tree of components $\mathbb{P}^1_\ell$.

Statements (ii) and (iii) are proved as in the previous proposition. One easily sees that all branch points have stabilizer C_{q+1}.

Statement (iv) follows from statement (ii) and the Riemann-Hurwitz formula. □

References

[A-B] Peter Abramenko and Kenneth S. Brown, *Buildings Theory and Applications*, Springer Verlag, 2008.

[B-E-K-S] Ron Baker, Gary L. Ebert, Gabor Korchmaros and Tamas Szonyi, *Orthogonally divergent spreads of Hermitian curves*, Finite Geometry and Combinatorics (Deinze, 1992), London Math. Soc. Lecture Note Ser. **191** (1993), Cambridge Univ. Press, 17–30.

[B-E] Susan Barwick and Gary Ebert, *Unitals in Projective Planes*, Springer Monographs in Mathematics, Springer Verlag, 2008.

[B-G-R] S. Bosch, U. Güntzer and R. Remmert, *Non-archimedean Analysis*, Springer Verlag, 1984.

[Dr] V.G. Drinfel'd, *Coverings of p-Adic Symmetric Regions*, Functional Anal. Appl. **10** (1976), 107-115.

[F-P] J. Fresnel and M. van der Put, *Rigid Analytic Geometry and its applications*, Progress in Math. **218**, Birkhäuser, 2004.

[H] D.W. Hoffmann, *On positive definite hermitian forms*, Manuscripta Math. **71** (1991), 399–429.

[Ki] Markus Kirschmer, *Definite quadratic and hermitian forms with small class number*, Habilitationsschrift, RWTH Aachen University, 2016.

[L-V] K.F. Lai and H. Voskuil, *P-adic automorphic functions for the unitary group in three variables*, Algebra Colloquium **7** (2000), 335–360.

[O-R] Sacha Orlik and Michael Rapoport, *Deligne-Lusztig varieties and period domains over finite fields*, Journal of Algebra **320** (2008), 1220–1234.

[P] Lue Pan, *First covering of Drinfel'd upper half-plane and Banach representations of $GL_2(\mathbb{Q}_p)$*, Algebra Number Theory **11** (2017), 405–503.

[P-V] M. van der Put and H. Voskuil, *Symmetric spaces associated to split algebraic groups over a local field*, J. Reine Angew. Math. **433** (1992), 69–100.

[S] A. Schiemann, *Classification of hermitian forms with the neighbourhood method*, J. Symb. Comput. **26** (1998), 487–508.

[Te] J. Teitelbaum, *Geometry of an étale covering of the p-adic upper half plane*, Ann. Inst. Fourier (Grenoble) **40** (1990), 69–78.

[Ti] Jacques Tits, *Reductive groups over local fields*, in *Automorphic Forms, Representations, and L-functions*, Proc. Sympos. Pure Math. **33** (1979), Amer. Math. Soc., Providence, R.I., 29–69.

[Vol] Inken Vollaard, *The supersingular locus of the Shimura variety for GU(1,s)*, Canad. J. Math. **62** (2010), 668–720.

[Vol-W] Inken Vollaard and Torsten Wedhorn, *The supersingular locus of the Shimura variety of GU(1,n-1) II*, Inventiones Math. **184** (2011), 591–627.

[V] H. Voskuil, *On the action of the unitary group on the projective plane over a local field*, J. Austral. Math. Soc. (Series A) **62** (1997), 371–397.